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Dispersion Relation for K Meson-Nucleon Scattering and its Application

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The dispersion relation is used for analyzing K-N scattering data to determine the type of the K-coupling and the magnitude of its coupling constant. The following assumptions are made:

i) Spin of K is zero.
ii) Λ and Σ have the same parities.
iii) K^+-p interaction is repulsive.
iv) K scattering by p at low energies is isotropic.

Then it is concluded: If K^--p interaction is repulsive, K-coupling is scalar. If K^--p is attractive, the coupling could be either scalar or pseudoscalar depending on the energy dependence of the K^+-p scattering cross section at low energies.

§ 1. Introduction

In the previous paper[1] it was pointed out that there is an essential difference between dispersion relation for K mesons with both types of interactions: scalar and pseudoscalar. Also discussion was given about the contributions to the scattering amplitude arising from the unphysical region, due to the exothermic reactions $K+p\rightarrow\pi+\Lambda$ and $K+p\rightarrow\pi+\Sigma$.

In the present paper it is examined if this dispersion relation determines the type of K coupling and the magnitude of its coupling constant, with a few data available now for K-p scattering.

The cross section of K^+-p elastic scattering appears to be energy-independent in the energy region 60 to 200 Mev.[2),3)] In the lower energy region, the propane bubble chamber result indicates that K^+-p scattering cross section may increase with energy[3),4)] while the emulsion data shows no appreciable energy-dependence.[2)] So that two curves are assumed for K^+-p scattering cross section in our analysis. The relative parity between Λ and Σ is assumed to be positive[5)] and K^+-p interaction is taken to be repulsive.[3),6)] We also accept that K scattering by p at low energies is isotropic.[2),3)] If the scattering amplitude in the unphysical region has neither a cusp nor kink, the smooth extrapolation into this region would be a good approximation, and its contribution proves to be fairly small.

With these data the dispersion relation can be applied to determine the K-coupling.

§ 2. Dispersion relations for K-p scattering

In the paper I the dispersion relations (I·5·1), (I·5·3) were discussed to be more convenient for the determination of K-coupling. (I·5·1) is

$$\omega\frac{D_-(\omega)-D_+(\omega)}{2}=\frac{2f'^2}{\omega^2-\omega_\Lambda{}^2}+\frac{2g'^2}{\omega^2-\omega_\Sigma{}^2}+\frac{\omega^2}{\pi}\int_{\omega_{\Lambda\pi}}^{m_k}d\omega'\frac{A_-(\omega')}{\omega'^2-\omega^2}$$

$$+\frac{\omega^2}{4\pi^2}\int_{m_k}^{\infty}d\omega' k'\left[\frac{\sigma_-(\omega')-\sigma_+(\omega')}{\omega'^2-\omega^2}\right] \qquad (2\cdot1)$$

where $f'^2,\ g'^2>0$ ps. case
<0 s. case

$$\omega_\Lambda=\frac{M_\Lambda{}^2-M_p{}^2-m_k{}^2}{2M_p}=0.107\ m_k$$

$$\omega_\Sigma=\frac{M_\Sigma{}^2-M_p{}^2-m_k{}^2}{2M_p}=0.268\ m_k \qquad (2\cdot2)$$

$$\omega_{\Lambda\pi}=\frac{(M_\Lambda+m_\pi)^2-M_p{}^2-m_k{}^2}{2M_p}=0.404\ m_k.$$

However, since the data on K^--p scattering is very poor, the following dispersion relation is used here. Neglecting $\omega_\Lambda{}^2$ and $\omega_\Sigma{}^2$ (since $\omega_\Lambda{}^2/m_k{}^2=1.14\times10^{-2}$, $\omega_\Sigma{}^2/m_k{}^2=7.18\times10^{-2}$) in the Born term we get a simplified expression.

$$D_+(\omega)-\tfrac{1}{2}(1+\omega)D_+(1)-\tfrac{1}{2}(1-\omega)D_-(1)$$

$$=\frac{k^2}{4\pi^2}\int_1^\infty\frac{d\omega'}{k'}\left[\frac{\sigma_+(\omega')}{\omega'-\omega}+\frac{\sigma_-^{sc}(\omega')+\sigma_-^{ab}(\omega')}{\omega'+\omega}\right]+\frac{k^2}{\pi}\int_{\omega_{\Lambda\pi}}^{1}d\omega'\frac{A_-(\omega')}{k'^2(\omega'+\omega)}+2F\frac{k^2}{\omega}. \qquad (2\cdot3)$$

Here K-meson mass is taken as unity,* $\sigma_-^{sc}(\omega)$ involves both K^--p elastic and charge exchange scattering cross section and

$$2F=-\frac{1}{2M_p}\frac{1}{M_\Lambda-M_p-\omega_\Lambda}f_{\Lambda k}^2-\frac{1}{2M_p}\frac{1}{M_\Sigma-M_p-\omega_\Sigma}f_{\Sigma k}^2$$

$$=-1.01\,f_{\Lambda k}^2-0.64\,f_{\Sigma k}^2 \qquad <0.\ \text{s. case} \qquad (2.4)$$

$$=\frac{1}{2M_p}\frac{1}{M_\Lambda+M_p+\omega_\Lambda}f_{\Lambda k}^2+\frac{1}{2M_p}\frac{1}{M_\Sigma+M_p+\omega_\Sigma}f_{\Sigma k}^2$$

$$=0.062\,f_{\Lambda k}^2+0.060\,f_{\Sigma k}^2 \qquad >0\ \text{ps. case.}$$

* Hereafter this unit will be adopted.

where $f_{\Lambda k}^2$, and $f_{\Sigma k}^2$ are the rationalized and renormalized ΛNK and ΣNK-coupling constants, respectively. If we rewrite eq. (2·3) as

$$\left[\frac{1}{k^2}\{D_+(\omega)-\tfrac{1}{2}(1+\omega)D_+(1)-\tfrac{1}{2}(1-\omega)D_-(1)\}\right.$$

$$\left.-\frac{1}{4\pi^2}\int_1^\infty \frac{d\omega'}{k'}\left\{\frac{\sigma_+(\omega')}{\omega'-\omega}+\frac{\sigma_-^{sc}(\omega')}{\omega'+\omega}+\frac{\sigma_-^{ab}(\omega')}{\omega'+\omega}\right\}-\frac{1}{\pi}\int_{\omega_{\Lambda\pi}}^1 d\omega'\frac{A_-(\omega')}{k'^2(\omega'+\omega)}\right]\omega$$

$$=2F, \qquad (2\cdot5)$$

then the apparently complicated ω dependence indicated by the left-hand side of (2·5) should give $2F$. If F is negative, the interaction is found to be of a scalar type, and the positive F means pseudoscalar coupling.

In this form the result is insensitive to $\sigma_-(\omega)$ and the choice of unknown $A_-(\omega)$ in the unphysical region. It is also insensitive to the behavior of $\sigma_+(\omega)$ at high energies, but sensitive to the low energy K^+-p scattering.

§ 3. Experimental data

As to the cross section of K^+-p scattering, the propane bubble chamber result indicates that K^+-p scattering cross section may increase with energy below 60 Mev,[3),4)] while the emulsion data show it to be energy-independent. Hence two curves (Case A, Case B) are assumed for $\sigma_+(\omega)$ below 200 Mev (see Fig. 1). Experimental data for the interaction of 250—300 Mev K^+ mesons with emulsion nuclei tell us that in this energy region the K-H cross section based on a very few events appears to be quite close to the value of 15 mb obtained at lower energies.[7)] So $\sigma_+(\omega)$ is assumed to be

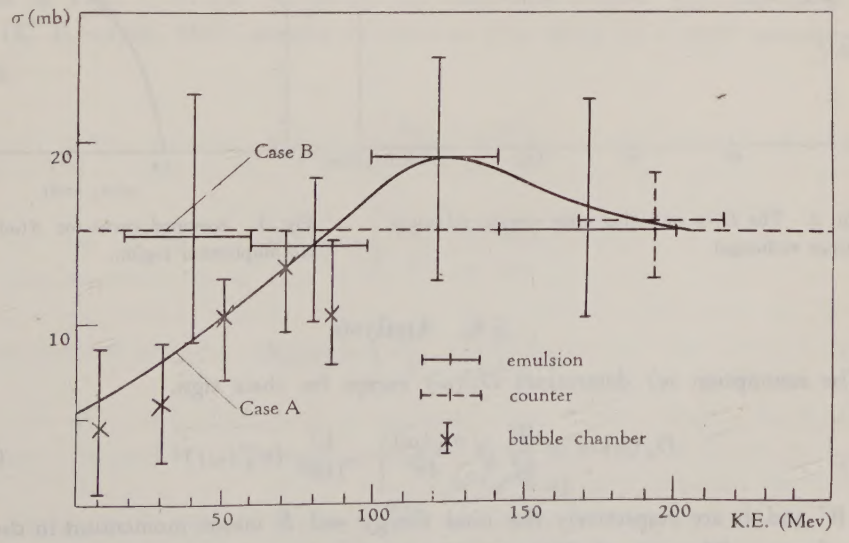

Fig. 1. The K^+-p cross section.

constant (=15 mb) above 200 Mev; however, this assumption is not so serious because the convergence of the integral (2·3) used here is very good.

There has been no analysis of K^--p scattering, elastic and charge exchange, comparable to what we have for K^+ just because it is too complicated. However, this is not so serious since in our dispersion relation (2·3) the contribution to the integral coming from K^--p scattering is fairly small owing to the large denominator. As to the charge exchange scattering, Ceccarelli[3] estimated a value of about 1 for the ratio of elastic K^--p scattering to the rest of the processes (charge exchange+absorption). If we accept this value, then K^--p scattering cross section $(=\sigma_-^{el}+\sigma_-^{c.e.})$ is given (see Fig. 2).

With respect to $\sigma_-^{ab}(\omega)$ we assume the $1/k$ dependence since it seems that K^- mesons are captured from S-states. $k\sigma_-^{ab}(\omega)$ then is put equal to about 7 (K meson mass unit)[3] below 200 Mev and smoothly extrapolated into the unphysical region (see Fig. 3). Contributions from this region, however, is rather small.

With these data together with the assumptions i)~iv) it will be analyzed if we can obtain some information about the K-coupling in § 4.

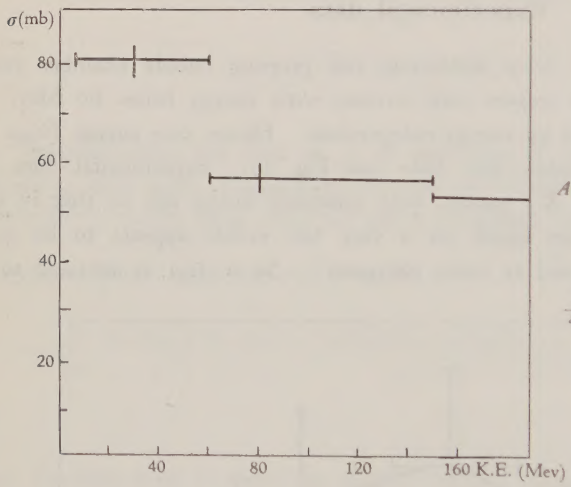

Fig. 2. The K^--p scattering cross section (elastic+charge exchange).

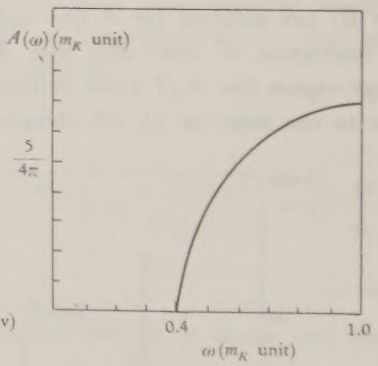

Fig. 3. Assumed curve for $A(\omega)$ in the unphysical region.

§ 4. Analysis

The assumption iv) determines $D_\pm(\omega)$ except for their sign.

$$D_\pm(\omega)=\pm\frac{W}{M_p}\sqrt{\frac{\sigma_\pm^{el}(\omega)}{4\pi}-\frac{k_b^2}{16\pi^2}(\sigma_\pm^{tot}(\omega))^2} \qquad (4\cdot1)$$

where W and k_b are respectively the total energy and K meson momentum in the center-of-mass frame. The sign of $D_+(\omega)$ is minus by the assumption iii). However, for

$D_-(\omega)$ the sign plus or minus depends on whether K^--p interaction is attractive or repulsive. $D_+(\omega)$ are calculated for Case A and Case B at two energies 50 Mev and 80 Mev and are listed in Table I.

Table I. Value of $D_+(\omega)$

K. E.	Case A	Case B
0 Mev ($\omega=1$)	-0.75	-1.33
50 (1.10)	-1.09	-1.32
80 (1.16)	-1.28	-1.31

Accepting the value of $k_b\sigma_-^{ab}$ to be 7, we get

$$D_-(1) = \pm 2.26. \tag{4·2}$$

Hence the first term on the left-hand side in eq. (2·5) can be evaluated. See Table II.

Table II. Value of $1/k^2\{D_+(\omega) - \frac{1}{2}(1+\omega)D_+(1) - \frac{1}{2}(1-\omega)D_-(1)\}$

K. E.	$1/k^2\{D_+(\omega) - \frac{1}{2}(1+\omega)D_+(1)\}$		$-(1/k^2)\frac{1}{2}(1-\omega)D_-(1)$	
	Case A	Case B	Case A	Case B
50 Mev ($\omega=1.10$)	-1.44	0.38	± 0.54	± 0.54
80 Mev ($\omega=1.16$)	-1.35	0.37	± 0.33	± 0.33

The integral in eq. (2·5) can also be numerically calculated with the cross section assumed in Figs. 1, 2 and 3. With this dispersion relation the contributions above $\omega\sim 4$ (K. E.$\sim$1500 Mev) amount to less than 5% owing to a rapid convergence. By defining

$$X_1(\omega) = \int_1^4 \frac{d\omega'}{k'} \frac{\sigma_+(\omega')}{\omega'-\omega}, \tag{4·3}$$

$$X_2(\omega) = \int_1^4 \frac{d\omega'}{k'} \frac{\sigma_-^{cc}(\omega')}{\omega'+\omega}, \tag{4·4}$$

$$X_3(\omega) = \int_1^4 \frac{d\omega'}{k'} \frac{\sigma_-^{ab}(\omega')}{\omega'+\omega}, \tag{4·5}$$

$$X'(\omega) = \int_{\omega_{\Lambda\pi}}^1 d\omega' \frac{A_-(\omega')}{k'^2(\omega'+\omega)}, \tag{4·6}$$

next table can be obtained for the Case A, B at 50 Mev and 80 Mev.

Table III. Value of the integral.

K. E.	$X_1(\omega)$	$X_2(\omega)$	$X_3(\omega)$	$4\pi X'(\omega)$	$(1/4\pi^2)\{X_1+X_2+X_3+4\pi X'\}$
50 Mev ($\omega=1.10$)	27.88 (A) −6.61 (B)	26.61	1.94	−6.27	1.27 (Case A) 0.40 (Case B)
80 Mev ($\omega=1.16$)	20.22 (A) −8.42 (B)	26.30	2.23	−5.81	1.09 (Case A) 0.37 (Case B)

Table III shows that the energy dependence of the K^+-p scattering cross section at low energies have a large effect on the dispersion integral.

$2F$ can be calculated with the aid of Tables II, III and eq. (2·5) and is listed in Table IV.

Table IV. Value of $2F$.

K. E.	Case A		Case B	
	K^-p		K^-p	
	attractive	repulsive	attractive	repulsive
50 Mev	−2.39	−3.58	0.57	−0.62
80 Mev	−2.45	−3.21	0.39	−0.37

§ 5. Conclusions and discussions

As was discussed in § 2, positive (negative) F means the pseudoscalar (scalar) coupling. By using the relation (2·4),

$$2F=-1.01\,f_{\Lambda k}^2-0.64\,f_{\Sigma k}^2 \quad \text{s. case}$$
$$=0.062\,f_{\Lambda k}^2+0.060\,f_{\Sigma k}^2 \quad \text{ps. case,}$$

the following conclusion can be derived. Repulsive K^-p interaction favors scalar type for K-baryon coupling, the coupling constants being the order of unity. This conclusion seems to be independent of the energy dependence of $\sigma_+(\omega)$.

For attractive K^--p interactions, the conclusion is dependent on the low energy behavior of $\sigma_+(\omega)$. When the low energy K^+-p cross section increases with energy (Case A), the scalar type is probable and its coupling constant is of the order of unity. However, if it remains constant (Case B), the pseudoscalar coupling is preferable, and the magnitude of its coupling constant is ~ 4.

The above analysis is valid even when $KK\pi\pi$ interaction exists. As the forward scattering amplitude arising from $KK\pi\pi$ interaction is energy independent, this one is dropped in the dispersion relation used here.

The author would like to express his deep gratitude to Dr. H. Miyazawa for his guidance and kind encouragement.

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Note added in proof.—The dispersion theorectical approach for determining the type of K-coupling and the magnitude of its coupling constant has also been given by P. T. Matthews and Abdus Salam (Phys. Rev. **110**, (1958), 569) and by Goebel (ibid. **110** (1958), 572), who have used essentially our eq. (2·1). In our analysis the conclusion is rather dependent on the low energy behavior of $\sigma^+(\omega)$ but insensitive to the high energy behavior, while in theirs the high energy behavior is more important than the low energy behavior.

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Hypothesis of Invariance of CP Alone and Conservation of Parity in Strong Interactions

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Conditions for the parity conservation in the strong interaction are worked out under the assumption of CP invariance only. The required conditions are the following: Either (i) the Fermi type interaction Hamiltonian (eqs. (1) and (2)) should be charge-independent and its coupling type should be an arbitrary linear combination of s, t and p. Or (ii) the coupling type of the interaction Hamiltonian should be an arbitrary linear combination of s–p and t.

§ 1. Introduction

According to recent experiments, the strong and electromagnetic interaction Hamiltonians are invariant under the charge conjugation (C), space reflection (P) and time reversal (T), while the weak interaction Hamiltonian is not invariant under C and P, though it is not yet clear whether it is invariant under T or not. On the other hand, the strong interaction Hamiltonian is invariant under the rotation in charge space, while the weak interaction Hamiltonian is not. So far we have not yet clarified any connection between the non-conservation of C and P, and the non-invariance under the rotation in charge space in the weak interaction. It is still an open question which of the two characteristics—violence of charge independence or that of C and P—should be regarded as the more fundamental.

For the purpose of finding out a clue to this question, we make the following consideration: Assuming that the strong, electro-magnetic and weak interaction Hamiltonians are invariant under combined transformation CP only, but not necessarily under the P- and C-transformation separately, we search for the condition that P and C are conserved in the strong interaction Hamiltonian.

§ 2. Strong interaction Hamiltonian

Fundamental interactions assumed in this note have the following form,

$$H' = (\bar{P}P)(\bar{P}P) + (\bar{N}N)(\bar{N}N) + (\bar{P}P)(\bar{N}N) + (\bar{\Lambda}\Lambda)(\bar{\Lambda}\Lambda) + (\bar{P}P)(\bar{\Lambda}\Lambda) + (\bar{N}N)(\bar{\Lambda}\Lambda) \tag{1}$$

$$H'' = (\bar{P}N)(\bar{N}P) \tag{2}$$

where the coupling constants of each term can be arbitrary, and

$$(\bar{A}B)\,(\bar{C}D) \equiv \sum_i g_i (\bar{\psi}_A O_i \psi_B)\,(\bar{\psi}_C O_i \psi_D) + \sum_i g'_i (\bar{\psi}_A O_i \psi_B)\,(\bar{\psi}_C O_i \gamma_5 \psi_D) + \text{H.C.} \quad (3)$$

$$\bar{\psi} = \psi^+ \gamma_4 \quad (4)$$

($^+$ means Hermitian conjugate.),

P, N and Λ representing proton, neutron and Λ-particle respectively. The symbol O_i corresponds to one of the well-known five possibilities of interaction scalar (s), pseudo-scalar (p), vector (v), axical-vector (a) and tensor (t).

When the interaction is charge independent, the coupling constants of these representations are related to each other and the interaction Hamiltonians have the following forms:

$$H_1 = \sum_i G_{1i} (\bar{\psi} O_i \tau_4 \psi)\,(\bar{\psi} O_i \tau_4 \psi) + \sum_i G'_{1i} (\bar{\psi} O_i \tau_4 \psi)\,(\bar{\psi} O_i \gamma_5 \tau_4 \psi) \quad (5)$$

$$H_2 = \sum_i \sum_{k=1}^{3} G_{2i} (\bar{\psi} O_i \tau_k \psi)\,(\bar{\psi} O_i \tau_k \psi) + \sum_i \sum_{k=1}^{3} G'_{2i} (\bar{\psi} O_i \tau_k \psi)\,(\bar{\psi} O_i \gamma_5 \tau_k \psi) \quad (6)$$

$$H_3 = \sum_i g_{3i} (\bar{\psi}_\Lambda O_i \psi_\Lambda)\,(\bar{\psi}_\Lambda O_i \psi_\Lambda) + \sum_i g'_{3i} (\bar{\psi}_\Lambda O_i \psi_\Lambda)\,(\bar{\psi}_\Lambda O_i \gamma_5 \psi_\Lambda) \quad (7)$$

$$H_4 = \sum_i g_{4i} (\bar{\psi} O_i \tau_4 \psi)\,(\bar{\psi}_\Lambda O_i \psi_\Lambda) + \sum_i g'_{4i} (\bar{\psi} O_i \tau_4 \psi)\,(\bar{\psi}_\Lambda O_i \gamma_5 \psi_\Lambda) \quad (8)$$

where

$$\psi = \begin{pmatrix} \psi_P \\ \psi_N \end{pmatrix}. \quad (9)$$

If we make use of the idea of Sakata's compound model[1] (i.e. $\pi = N + \bar{N}$, $\Sigma = \Lambda + N + \bar{N}$, $K = \bar{\Lambda} + N$ and $\Xi = \Lambda + N + \bar{N}$ (where N stands for proton or neutron), which means that a π-meson is considered as a compound particle composed of nucleon antinucleon, and so on), the above interactions will yield interactions between baryons and mesons, although properties of these interaction must be studied in more detail.

§ 3. Condition of parity conservation in strong interaction

First we can prove the following theorem.

[Theorem 1] The parity non-conserving Hamiltonian of the type

$$H = \sum_i g'_i (\bar{\psi}_A O_i \psi_A)\,(\bar{\psi}_B O_i \gamma_5 \psi_B) \quad (10)$$

is not invariant under the combined transformation CP, if O_i represents s, p or t; but it can be CP-invariant, if O_i represents v or a.

(Proof) For the spin 1/2 field $\psi(x)$, the CP transformation is

$$\psi'(x) = S^{-1} \psi^+ (-\boldsymbol{x}, x_4), \qquad \psi'^+ (x) = \psi(-\boldsymbol{x}, x_4) S \quad (11)$$

where

$$S \gamma_\mu S^{-1} = -\gamma_\mu^T. \quad (12)$$

(^{T} means transposed.)

The expression $\bar{\psi}_A O_i \psi_B$ in the Hamiltonian should be properly antisymmetrized as

$$\tfrac{1}{2}(\bar{\psi}_A O_i \psi_B - \psi_B O_i^T \bar{\psi}_A). \tag{13}$$

So the term $\bar{\psi}_A O_i \psi_A$ is transformed under CP to

$$\begin{aligned}\bar{\psi}'_A O_i \psi'_A &\sim \tfrac{1}{2}(\bar{\psi}'_A O_i \psi'_A - \psi'_A O_i^T \bar{\psi}'_A) \\ &= \tfrac{1}{2}(\psi_A S \gamma_4 O_i S^{-1} \psi_A^+ - \psi_A^+ S^{-1T} O_i^T \gamma_4^T S^T \psi_A) \\ &= \pm \tfrac{1}{2}(\psi_A O_i^T \bar{\psi}_A - \bar{\psi}_A O_i \psi_A)\end{aligned} \tag{14}$$

where the sign of the last expression depends on the form of O_i. If the expression $\bar{\psi}_A O_i \psi_A$ changes the sign under CP transformation and $\bar{\psi}_B O_i \gamma_5 \psi_B$ does not change its sign, the parity non-conserving interaction Hamiltonian (5) cannot be invariant under CP. This is the case when O_i represents s, p or t.

Theorem 1 can be applied to the expression (1) but not to (2). If we assume charge independence, however, the coupling constant of the expression (2) is related to that of the expression (1), as indicated in the expression (6), and we get the following theorem.

[Theorem 2] When the interaction Hamiltonian is charge independent and has the type of the expressions (5) (6) (7) and (8),* parity non-conserving terms cannot be invariant under the transformation CP,** if O_i represents an arbitrary linear combination of s, p and t;*** but they can be CP invariant, if O_i represents an arbitrary linear combination of v and a.

From this theorem we get the condition that P and C are conserved separately in the strong interaction Hamiltonian. In this theorem we required that the interaction Hamiltonian should be charge independent in order to conserve parity in the strong interaction.

But we can also show that, even without charge independence hypothesis, parity non-conserving terms cannot appear in the interaction Hamiltonian if a particular linear combination of s, t and p terms is chosen. For this purpose the following formula is useful.

[Formula] The most general form of the direct coupling Fermi interaction Hamiltonian is the linear combination of the following terms,

---

* For this purpose it is not necessary to assume that the interaction term $(\bar{N}N)(\bar{\Lambda}\Lambda)$ is charge independent.

** Soloviev[2] restricts the interaction Hamiltonian to the Yukawa type and direct pseudo-scalar coupling, so he has not been able to forbid parity non-conserving strong interaction of K-mesons.

*** The representations (5) and (6) are not independent of each other when O_i corresponds to all of the five possibilities, s, p, t, v and a. So, even if the type of representations (5) and (6) contains v and a, there are certain cases in which these representations can be rewritten in the form that the type of representations (5) and (6) corresponds only to an arbitrary linear combination of s, p and t.

$$
\begin{aligned}
H_1 &= (\bar{\Phi}\Psi)\,(\bar{\varphi}\phi), \\
H_2 &= (\bar{\Phi} i\gamma_\mu \Psi)\,(\bar{\varphi} i\gamma_\mu \phi), \\
H_3 &= \Big(\bar{\Phi}\,\frac{i}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\,\Psi\Big)\Big(\bar{\varphi}\,\frac{i}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\,\phi\Big), \\
H_4 &= (\bar{\Phi} i\gamma_\mu\gamma_5 \Psi)\,(\bar{\varphi} i\gamma_\mu\gamma_5 \phi), \\
H_5 &= (\bar{\Phi} i\gamma_5 \Psi)\,(\bar{\varphi} i\gamma_5 \phi),
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
K_1 &= (\bar{\Phi}\Psi)\,(\bar{\varphi}\gamma_5\phi), \\
K_2 &= (\bar{\Phi} i\gamma_\mu \Psi)\,(\bar{\varphi} i\gamma_\mu\gamma_5 \phi), \\
K_3 &= \Big(\bar{\Phi}\,\frac{i}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\,\Psi\Big)\Big(\bar{\varphi}\,\frac{i}{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)\gamma_5\phi\Big), \\
K_4 &= (\bar{\Phi} i\gamma_\mu\gamma_5 \Psi)\,(\bar{\varphi} i\gamma_\mu \phi), \\
K_5 &= (\bar{\Phi} i\gamma_5 \Psi)\,(\bar{\varphi} i\phi).
\end{aligned}
\tag{16}
$$

If we now exchange Φ and φ and define H_i' and K_i' by

$$
\begin{aligned}
H_1' &= (\bar{\varphi}\Psi)\,(\bar{\Phi}\phi), \text{ etc.} \\
K_1' &= (\bar{\varphi}\Psi)\,(\bar{\Phi}\gamma_5\phi), \text{ etc.,}
\end{aligned}
\tag{17}
$$

then H_i' can be written as the linear combination of H_i's as follows :[3]

$$
\begin{aligned}
4H_1' &= -H_1 + H_2 - H_3 - H_4 + H_5, \\
4H_2' &= 4H_1 + 2H_2 + 2H_4 + 4H_5, \\
4H_3' &= -6H_1 + 2H_3 + 6H_5, \\
4H_4' &= -4H_1 + 2H_2 + 2H_4 - 4H_5, \\
4H_5' &= H_1 + H_2 + H_3 - H_4 - H_5.
\end{aligned}
\tag{18}
$$

If we replace H_i's by K_i's we get the equations for K_i'.

From this formula we see that when we rewrite the interaction Hamiltonian $(\bar{N}P)\,(\bar{P}N)$ in the form of $(\bar{N}N)\,(\bar{P}P)$, the coupling type of $(\bar{N}N)\,(\bar{P}P)$ is also a linear combination of s, t and p, if the type of $(\bar{N}P)\,(\bar{P}N)$ is an arbitrary linear combination of (s-p) and t. So we get the following theorem.

[Theorem 3] When the interaction Hamiltonian is written in the form of equations (1) and (2), and the type of coupling is an arbitrary linear combination of s-p and t,* no term can be present that would be CP invariant and would not conserve parity.

This theorem is another condition which requires that P and C are conserved separately in the strong interaction Hamiltonian.

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* The coupling type s-p-t is invariant under the exchange of wave function Ψ and ϕ.

§ 4. Conclusion and remarks

Summarizing, we can say: In this note we assumed that the strong, electromagnetic and weak interaction Hamiltonians are invariant under $CP=T$ transformation only. The strong and electromagnetic interaction Hamiltonians, however, conserve parity, if there are the following conditions:

(i) In the strong interaction, if the interaction Hamiltonian is charge independent and the type of coupling is an arbitrary linear combination of s, p and t, provided that the fundamental interaction is assumed to be the form of eqs. (1) and (2); or if the type of coupling is an arbitrary linear combination of s-p and t.

(ii) In the electromagnetic interaction, if the interaction is gauge invariant.[2]

It is very important to examine what coupling types appear in the Yukawa type interaction $\bar{N}N\pi$ and $\bar{\Lambda}NK$ when we assume that they are due to the more fundamental Fermi interaction (through the composite model) and further restrict the coupling type as in theorems 2 and 3. We do not enter into this problem here, but we should like to point out only the possibility that the coupling type of π and that of K are different, as can be seen from the representations (5)—(8) and Formula.

The point of view adopted in this note is the following. We do not regard parity conservation or violation as a fundamental principle concerning the interaction of elementary particles: Parity happens to be conserved in strong and electromagnetic interactions but not in the weak interactions. The former cases can take place only when the number of each kind of particles participating in the Fermi interaction does not change. (Notice that we are discussing the interaction based on the Sakata model of composite particles with which the fundamental constituent particles (N and Λ) can change only through weak interactions.) Charge independence is valid under the same condition, so that charge independence and parity conservation may have some connection. We have a feeling, however, that the connection between parity and charge space or strangeness is, if any, relatively loose, as is suggested by the present analysis. It seems to us that what is essential in the classification of interactions into strong and weak ones is not whether the interaction conserves parity, but whether fundamental particles change their nature: The strong interaction happens to conserve parity and strangeness, for the fundamental particles do not change their nature. And the weak interaction can neither forbid parity non-conserving terms nor conserve strangeness, for the fundamental particles change their nature ($N \leftrightarrow \Lambda$).

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Quantum Statistical Mechanics of Electron-Phonon System, II

——Specific Heat and Spin Paramagnetic Susceptibility——

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The effect of the electron-phonon interaction on specific heat and spin paramagnetic susceptibility of conduction electrons in metals are investigated using the method proposed by the author in the previous paper. Calculations up to terms of the second order of the interaction Hamiltonian show that specific heat is increased and spin paramagnetic susceptibility is decreased by the presence of the electron-phonon interaction, and the effect is much larger in the former than in the latter.

§ 1. Introduction

In the previous report,[1]* a statistical mechanical treatment was developed concerning the interaction between a system of conduction electrons in metals and quantized thermal vibrations of the ionic lattice (phonons). The treatment given there was limited mainly to the clarification of roles of various terms which appeared in the calculation up to terms of the second order of the interaction Hamiltonian. In this second report, the effects of the interaction on the electronic specific heat γT and the spin paramagnetic susceptibility χ_s are investigated in detail following the method developed in [I]. The problem of the electronic specific heat has already been dealt with by Zimmermann,[2] Buckingham-Schafroth[3] and Kaschluhn[4] from a standpoint similar to ours. Among their results, however, there are some discrepancies which may be the consequence of the differences in their methods of approximation. Unfortunately, we cannot decide here which of their results is the most appropriate one, but to present the result of our treatment which is based upon a straightforward perturbation expansion will serve to remove these uncertainities to some extent. Our result agrees essentially with that of Zimmermanns'.

The inter-electronic interaction which is taken into consideration in this paper is the effective electron-electron interaction induced by the electron-phonon interaction. The Coulomb interaction between electrons is neglected in the present stage. Because of this simplification, our results cannot be compared directly with experiments. It will also be noted here that the theoretical values of γ and χ_s which are given by Pines' theory might not be satisfactory either, since in his treatment only the Coulomb interaction is taken into account and the effective electron-electron interaction which will be investigated here is neglected. To investigate both effects at the same time a careful treatment of

* We refer it as [I].

cross terms will be necessary. As pointed out by Bardeen and Pines,[9] we must revise the interaction Hamiltonian itself used in the present treatment, considering the Coulomb interaction.

Specific Heat of the Electron System

§ 2. Fundamental equations

The total Hamiltonian of our electron-phonon system is given by ([I] (2·2))

$$\mathcal{H}=\mathcal{H}_0+\mathcal{H}_I+\mathcal{H}_R$$

$$\mathcal{H}_0=\sum_k \varepsilon_k a_k^+ a_k+\sum_q \hbar s q\, b_q^+ b_q$$

$$\varepsilon_k=\frac{\hbar^2}{2m}k^2,\quad k=|\boldsymbol{k}|,\quad q=|\boldsymbol{q}|$$

$$\mathcal{H}_I=-i\Gamma_0\sum_k\sum_q \sqrt{q}\,(b_q a_{k+q}^+ a_k-b_q^+ a_{k-q}^+ a_k) \tag{2·1}$$

$$\Gamma_0^2=4F_0\zeta_0\hbar s_0/3N_A$$

$$F_0=3C_I^2/8\zeta_0 M s_0^2,\quad \zeta_0=\frac{\hbar^2}{2m}k_0^2,\quad k_0=2\pi(3N/8\pi V)^{1/3}$$

$$\mathcal{H}_R=\alpha\sum_k \varepsilon_k a_k^+ a_k+\hbar(s_0-s)\sum_q q\, b_q^+ b_q$$

where $\mathcal{H}_0$ is the Hamiltonian of unperturbed electrons and phonons; $\mathcal{H}_I$ is the Bloch-Bethe's interaction Hamiltonian between electrons and phonons; $\mathcal{H}_R$ is introduced to renormalize the electron mass and the sound velocity. Notations in these expressions are the same as in [I], except for slight alterations.

Using this Hamiltonian, we define the grand partition function Ξ by

$$\Xi=\mathrm{Sp}\,\{\exp(\beta\zeta\textstyle\sum_r a_r^+ a_r)\exp(-\beta\mathcal{H})\}.$$

Expanding the operator $e^{-\beta\mathcal{H}}$ in powers of the operator $\mathcal{H}_I+\mathcal{H}_R$, we can rearrange our grand partition function in a form

$$\Xi=\Xi_0\exp\Big(\sum_{i=1}^{\infty}C_i\Big),$$

where Ξ_0 is the grand partition function of unperturbed electrons and phonons. In the approximation to the second order of $\mathcal{H}_I+\mathcal{H}_R$, we get the grand partition function $\Xi^{(e)}$ of the electron system in a following form, ([I]—(2·1), (2·4))

$$\Xi^{(e)}=\Xi_0^{(e)}\exp(\beta^2 A_2)$$

$$\Xi_0^{(e)}=\prod_k\{1+\exp\beta(\zeta-\varepsilon_k)\}$$

[9] The notation μ was used in [I] for the chemical potential of the electron following Prof. Mayer's text book. To avoid confusion with μ_0, which will be used for the Bohr magneton, we will use the notation ζ in this paper in place of μ. The notation of the effective mass m^* in [I] is replaced by m for the sake of simplicity while m_0 is used for the true electron mass.

$$\beta A_2 = \frac{4m\Gamma_0^{\prime 2}\sigma}{\hbar^2} \sum_k \sum_q \Phi(k,\ k')\, f_k f_{k'} \qquad (2\cdot 2)^*$$

$$f_k = \frac{1}{1+\exp\{\beta(\varepsilon_k - \zeta)\}}$$

$$\Phi(k,\ k') = \frac{q^2}{(k'^2 - k^2)^2 - \sigma^2 q^2} \qquad (2\cdot 3)$$

$$q = |k' - k| \leq q_m, \quad \sigma = 2ms/\hbar .$$

The phonon part has been separated by making use of the renormalization conditions for the sound velocity s and the effective mass m ([I]—(4·2) and (4·3)), given by

$$-2\beta\alpha \sum_k \varepsilon_k f_k + \beta \frac{4m\Gamma_0^{\prime 2}}{\hbar^2} \sum_k \sum_q \frac{q}{k'^2 - k^2 + \sigma q} f_k = 0 \qquad (2\cdot 4)$$

$$-\beta\hbar(s_0 - s) \sum_q q\bar{N}_q + \beta \frac{8m\Gamma_0^{\prime 2}}{\hbar^2} \sum_k \sum_q \frac{q(k'^2 - k^2)}{(k'^2 - k^2)^2 - \sigma^2 q^2} \bar{N}_q f_k = 0. \qquad (2\cdot 5)$$

The Helmholtz free energy Φ of the electron system is obtained by the relations

$$\Phi = N\zeta - \kappa T \log \Xi^{(e)} \qquad (2\cdot 6)$$

$$N = \kappa T \frac{\partial}{\partial \zeta} \log \Xi^{(e)}. \qquad (2\cdot 7)$$

Hence the specific heat C_V of the electron system is given by

$$C_V = -T\left(\frac{\partial^2 \Phi}{\partial T^2}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V = T\left(\frac{\partial}{\partial T}\left(\frac{\partial}{\partial T}\ \kappa T \log \Xi^{(e)}\right)_{\zeta, V}\right)_V , \qquad (2\cdot 8)$$

where S is the entropy and $(\partial X/\partial T)_{\zeta, V}$ means to differentiate X with respect to T under constant ζ and V. In carrying out the above mentioned differentiation, the temperature dependence of m, s and $\beta^2 A_2$ must be considered. The evaluation of these quantities at 0°K given in [I] is insufficient for this purpose. Of these three quantities, m and s can easily be shown to have temperature dependence given by a factor such as

$$(1 + a(\kappa T/\zeta_0)^2 + a'(\kappa T/\zeta_0)^4 + \cdots),$$

where a's are numerical constants. Then we know that their contributions to C_V are of the order $F(\sigma/k_0)$ and are very small compared with that of $\beta^2 A_2$ as will be shown later. Hence they can be neglected in eq. (2·8), that is, m and s may be treated as constants given in [I].

Eq. (2·8) becomes

$$C_V = C_V^{(0)} + C_V^{(1)}$$

* In the present paper, the summation $\sum_k$ which will be used hereafter does not contain the spin summation in contrast to that of [I].

$$C_V^{(0)}=N\left(\frac{\pi^2\kappa^2}{2\zeta_0}T-\frac{3\pi^4}{20}\frac{\kappa^4}{\zeta_0^3}T^3\cdots\right) \tag{2·9}$$

$$C_V^{(1)}=T\left\{\frac{\partial}{\partial T}\left(\frac{\partial}{\partial T}\beta A_2\right)_{\zeta,V}\right\}_V+\left(\frac{\partial\zeta}{\partial T}\right)\left\{\frac{\partial}{\partial\zeta}\left(\frac{\partial}{\partial T}\beta A_2\right)_{\zeta,V}\right\}_T,$$

where $C_V^{(0)}$ is the specific heat of the usual free electron gas. Modifications due to the shift* of ζ_0 by the presence of the interaction can also be neglected. The evaluation of the additional term $C_V^{(1)}$ will be given in the next section.

§ 3. Evaluation of $C_V^{(1)}$

The quantity βA_2 in (2·9) or in (2·2) can be written as

$$\beta A_2=\frac{4\zeta_0 F\sigma^2}{3N_A}I_2$$

$$I_2=2\Omega^2\int d\boldsymbol{k}\int d\boldsymbol{k}'\,\Phi(\boldsymbol{k},\ \boldsymbol{k}')f_{\boldsymbol{k}}f_{\boldsymbol{k}'} \tag{3·1}$$

$$\Omega=V/(2\pi)^3,\quad q=|\boldsymbol{k}'-\boldsymbol{k}|\leq q_m,$$

where the summation $2\sum_{\boldsymbol{k}}\sum_{\boldsymbol{q}}$ has been replaced by the integral $2\Omega^2\int d\boldsymbol{k}\int d\boldsymbol{k}'$, and $F\sigma$ is substituted for $F_0\sigma_0$ as in [I]. Hence the formula for $C_V^{(1)}$ is given in the following form, taking m and s for constants as has been mentioned in the previous section.

$$C_V^{(1)}=\frac{8\nu\zeta_0 F\sigma^2\Omega^2}{3N}\int d\boldsymbol{k}\int d\boldsymbol{k}'\,\Phi(\boldsymbol{k},\ \boldsymbol{k}')\left(-\frac{\partial}{\partial T}\left(\frac{\partial}{\partial T}f_{\boldsymbol{k}}f_{\boldsymbol{k}'}\right)_{\zeta,V}\right)_V$$
$$q=|\boldsymbol{k}'-\boldsymbol{k}|\leq q_m$$

Then we have

$$C_V^{(1)}=\frac{8\nu\zeta_0 F\sigma^2\Omega^2}{3N}\left\{\frac{1}{T}(L_1+L_2+L_3)+\beta\frac{\partial\zeta}{\partial T}(L_1'+L_2'+L_3')\right\}, \tag{3·2}$$

where

$$L_1=2\int d\boldsymbol{k}\int d\boldsymbol{k}'\,\Phi(\boldsymbol{k},\ \boldsymbol{k}')\beta^2(\varepsilon_{\boldsymbol{k}}-\zeta)(\varepsilon_{\boldsymbol{k}'}-\zeta)g_{\boldsymbol{k}}g_{\boldsymbol{k}'}$$
$$(|\boldsymbol{k}'-\boldsymbol{k}|\leq q_m)$$

$$L_2=2\int d\boldsymbol{k}\int d\boldsymbol{k}'\,\Phi(\boldsymbol{k},\ \boldsymbol{k}')\beta^2(\varepsilon_{\boldsymbol{k}}-\zeta)^2h_{\boldsymbol{k}}f_{\boldsymbol{k}'} \tag{3·3}$$
$$(\quad\ \ 〃\quad\ \ )$$

$$L_3=-4\int d\boldsymbol{k}\int d\boldsymbol{k}'\,\Phi(\boldsymbol{k},\ \boldsymbol{k}')\beta(\varepsilon_{\boldsymbol{k}}-\zeta)g_{\boldsymbol{k}}f_{\boldsymbol{k}'}$$
$$(\quad\ \ 〃\quad\ \ )$$

* Apart from the change of ζ_0 from the ideal gas value through the renormalized effective mass of electron.

$$L_1' = 2\int d\boldsymbol{k}\int d\boldsymbol{k}'\,\Phi(\boldsymbol{k},\ \boldsymbol{k}')\,\beta(\varepsilon_k-\zeta)\,g_k g_{k'}$$
$$(|\boldsymbol{k}'-\boldsymbol{k}|\leq q_m)$$

$$L_2' = 2\int d\boldsymbol{k}\int d\boldsymbol{k}'\,\Phi(\boldsymbol{k},\ \boldsymbol{k}')\,\beta(\varepsilon_k-\zeta)\,h_k f_{k'} \qquad (3\cdot4)$$
$$(\quad\ \ //\quad\ \ )$$

$$L_3' = -2\int d\boldsymbol{k}\int d\boldsymbol{k}'\,\Phi(\boldsymbol{k},\ \boldsymbol{k}')\,g_k f_{k'}$$
$$(\quad\ \ //\quad\ \ )$$

and

$$g_k = \frac{\partial}{\partial(\beta\zeta)} f_k = \frac{\exp[\beta(\varepsilon_k-\zeta)]}{(1+\exp[\beta(\varepsilon_k-\zeta)])^2} \qquad (3\cdot5)$$

$$h_k = \frac{\partial^2}{\partial(\beta\zeta)^2} f_k = \frac{\exp[\beta(\varepsilon_k-\zeta)](\exp[\beta(\varepsilon_k-\zeta)]-1)}{(1+\exp[\beta(\varepsilon_k-\zeta)])^3}.$$

Evaluations of these integrals are given in the appendix I. The main contributions come from L_2, L_3 and L_1. Retaining only the largest terms of the order T^1 and T^3, we have

$$C_V^{(1)} = N\frac{\pi^2\kappa^2}{2\zeta_0}\cdot\frac{1}{2}(4\nu)^{1/3}F\cdot T$$
$$+N\frac{\pi^4\kappa^4}{\zeta_0\kappa\theta_D}(4\nu)^{1/3}F\cdot\left(\frac{21}{30}\log\left(\frac{\theta_D}{T}\right)-\frac{3}{2\pi^4}l_4\right)T^3. \qquad (3\cdot6)$$

The first term gives the correction to Sommerfeld's specific heat coefficient $\gamma^{(0)}$ and the second term shows the appearance of a new T^3 term. This T^3 term is larger than the ideal gas term $-3N\pi^4\kappa^4T^3/20\zeta_0^3$ by a factor of the order $(\zeta_0/\kappa\theta_D)^2$, but it is small by a factor of the order $\kappa\theta_D/\zeta_0$ compared with the lattice specific heat.

Then the Sommerfeld formula is modified as follows.

$$C_V = (\gamma^{(0)}+\gamma^{(1)})\,T$$
$$\gamma^{(0)} = N\frac{\pi^2\kappa^2}{2\zeta_0} \qquad (3\cdot7)$$
$$\gamma^{(1)} = \gamma^{(0)}\cdot\frac{1}{2}(4\nu)^{1/3}F.$$

If it is allowed to assume the convergence of our procedure, it can be said that the electron-phonon interaction increases the electronic specific heat without destroying its linear dependence on temperature in the ideal gas case. Or, in the language of the one-electron approximation, the electron-electron interaction induced by the electron-phonon interaction increases the effective number of electrons which take part in thermal excitations. The result

$$\gamma^{(0)}T\left(1+\frac{1}{2}(4\nu)^{1/3}F\right)$$

coincides essentially with that of Zimmermann's

$$\gamma^{(0)}T\Big/\left(1-\frac{1}{2}(4\nu)^{1/3}F\right).$$

Numerical results will be discussed in § 6.

Spin Paramagnetism

§ 4. Fundamental equations

The general treatment of the interacting electron gas in a magnetic field will be a very complicated problem. Here we consider only how the spin paramagnetism of the electron gas is influenced by the electron-phonon interaction.

When the homogeneous magnetic field H is applied in the z-direction, the total Hamiltonian of our electron-phonon system becomes, by neglecting the orbital magnetic part,

$$\begin{aligned}&\mathcal{H}=\mathcal{H}_0+\mathcal{H}_I+\mathcal{H}_R\\&\mathcal{H}_0=\sum_r \varepsilon_r a_r^+ a_r+\sum_q \hbar s q\, b_q^+ b_q \qquad (4\cdot1)\\&\varepsilon_r=\frac{\hbar^2}{2m}k^2+\mu_0\sigma_z H,\quad r=(\boldsymbol{k},\ \sigma_z),\quad \mu_0=e\hbar/2m_0c,\end{aligned}$$

where σ_z is the z-component of the spin angular momentum of an electron in the unit of $\hbar/2$ and takes the value 1 or -1; $\mathcal{H}_I$ and $\mathcal{H}_R$ are the same as in (2·1).

Now, from the general formula for the grand partition function Ξ of the total system,

$$\Xi=\mathrm{Sp}\{\exp(\beta\zeta\textstyle\sum a_r^+ a_r)\exp(-\beta\mathcal{H})\}, \qquad (4\cdot2)$$

the grand partition function of the electron system $\Xi^{(e)}$ can be derived by a procedure similar to the one used in paper [I].

$$\Xi^{(e)}=\Xi_0^{(e)}\exp(-\beta R_0-\beta R_1+\beta^2(A_0+A_1+A_2)) \qquad (4\cdot3)$$

where

$$\begin{aligned}\Xi_0^{(e)}=&\prod_k(1+\exp[\beta(\zeta-\varepsilon_k-\mu_0H)])\qquad (4\cdot4)\\&\times(1+\exp[\beta(\zeta-\varepsilon_k+\mu_0H)])\end{aligned}$$

$$\begin{aligned}&R_0=\alpha\sum_k\varepsilon_k(f_k^+ +f_k^-)\\&R_1=\hbar(s_0-s)\sum_q q\bar{N}_q\end{aligned} \qquad (4\cdot5)$$

$$\beta A_0=\frac{2m\Gamma_0^2}{\hbar^2}\sum_k\sum_q\frac{q}{k'^2-k^2+\sigma q}(f_k^+ +f_k^-)$$

$$\beta A_1=\frac{2m\Gamma_0^{\prime 2}}{\hbar^2}\sum_{\boldsymbol{k}}\sum_{q}\frac{q(\boldsymbol{k}'^2-\boldsymbol{k}^2)}{(\boldsymbol{k}'^2-\boldsymbol{k}^2)^2-\sigma^2q^2}\bar{N}_q(f_{\boldsymbol{k}}^{+}+f_{\boldsymbol{k}}^{-}) \tag{4·6}$$

$$\beta A_2=\frac{2m\Gamma_0^{\prime 2}\sigma}{\hbar^2}\sum_{\boldsymbol{k}}\sum_{q}\frac{q^2}{(\boldsymbol{k}'^2-\boldsymbol{k}^2)^2-\sigma^2q^2}(f_{\boldsymbol{k}}^{+}f_{\boldsymbol{k}'}^{+}+f_{\boldsymbol{k}}^{-}f_{\boldsymbol{k}'}^{-})$$

and

$$f_{\boldsymbol{k}}^{\pm}=\frac{1}{1+\exp[\beta(\varepsilon_{\boldsymbol{k}}\pm\mu_0H-\zeta)]}. \tag{4·7}$$

Making use of the Taylor expansion of $f_{\boldsymbol{k}}^{\pm}$,

$$f_{\boldsymbol{k}}^{\pm}=f_{\boldsymbol{k}}\mp(\beta\mu_0H)g_{\boldsymbol{k}}+\frac{1}{2}(\beta\mu_0H)^2h_{\boldsymbol{k}}\cdots, \tag{4·8}$$

we can divide the above quantities as follows:

$$R_0=R_0^{(0)}+R_0^{(H)},$$
$$A_0=A_0^{(0)}+A_0^{(H)},\ \text{etc.},$$

where $R_0^{(0)}$, $A_0^{(0)}$, etc., are the ones at zero magnetic field and coincide with R_0, A_0, etc., given in paper [I]. $R_0^{(H)}$, $A_0^{(H)}$, etc., are given by

$$R_0^{(H)}=\left(\frac{\mu_0H}{\zeta}\right)^2\alpha(\beta\zeta)^2\sum_{\boldsymbol{k}}\varepsilon_{\boldsymbol{k}}h_{\boldsymbol{k}}$$

$$\beta A_0^{(H)}=\left(\frac{\mu_0H}{\zeta}\right)^2G(\beta\zeta)^2\sum_{\boldsymbol{k}}\sum_{q}\frac{q}{\boldsymbol{k}'^2-\boldsymbol{k}^2+\sigma q}h_{\boldsymbol{k}} \tag{4·9}$$

$$\beta A_1^{(H)}=\left(\frac{\mu_0H}{\zeta}\right)^2G(\beta\zeta)^2\sum_{\boldsymbol{k}}\sum_{q}\frac{q(\boldsymbol{k}'^2-\boldsymbol{k}^2)}{(\boldsymbol{k}'^2-\boldsymbol{k}^2)^2-\sigma^2q^2}\bar{N}_qh_{\boldsymbol{k}}$$

$$\beta A_2^{(H)}=\left(\frac{\mu_0H}{\zeta}\right)^2G\sigma(\beta\zeta)^2\sum_{\boldsymbol{k}}\sum_{q}\frac{q^2}{(\boldsymbol{k}'^2-\boldsymbol{k}^2)^2-\sigma^2q^2}(f_{\boldsymbol{k}}h_{\boldsymbol{k}'}+h_{\boldsymbol{k}}f_{\boldsymbol{k}'}+2g_{\boldsymbol{k}}g_{\boldsymbol{k}'})$$

$$G=4\nu F\sigma\zeta_0/3N.$$

Here we impose the same renormalization condition as in the case of zero magnetic field, i.e. eqs. (2·4) and (2·5) which in the present notations may be written as

$$-\beta R_0^{(0)}+\beta^2A_0^{(0)}=0$$
$$-\beta R_1+\beta^2A_1^{(0)}=0. \tag{4·10}$$

The choice of this renormalization condition is arbitrary in principle. We have chosen the above one to avoid m and s becoming H dependent.

Hence the grand partition function $\Xi^{(s)}$ becomes

$$\Xi^{(s)}=\Xi_0^{(s)}\exp[-\beta R_0^{(H)}+\beta^2(A_0^{(H)}+A_1^{(H)}+A_2^{(H)})+\beta^2A_2^{(0)}] \tag{4·11}$$

with the same values of m and s as in § 2.

The first and the second term in the exponential of eq. (4·11) are the additional terms coming from the influence of magnetic field.

The magnetic moment of the electron system is obtained by the equation

$$M=-\left(\frac{\partial \Phi}{\partial H}\right)_T=-\left(\frac{\partial}{\partial H}\left[N_\zeta^e-\kappa T\log \Xi^{(e)}\right]\right)_{\zeta,T} \tag{4·12}$$

$$=\kappa T\left(\frac{\partial}{\partial H}\log \Xi^{(e)}\right)_{\zeta,T}.$$

Hence we have

$$M=M^{(0)}+M^{(1)} \tag{4·13}$$

$$M^{(0)}=\kappa T\left(\frac{\partial}{\partial H}\log \Xi_0^{(e)}\right)_{\zeta,T}$$

$$M^{(1)}=\left(\frac{\partial}{\partial H}\left[-R_0^{(H)}+\beta A_0^{(H)}+\beta A_1^{(H)}+\beta A_2^{(H)}\right]\right)_{\zeta,T}. \tag{4·14}$$

In these formulas $(\partial x/\partial H)_{\zeta,T}$ means to differentiate x with respect to H under constant ζ and T. Then the susceptibility defined by

$$\chi_s=M/VH \tag{4·15}$$

becomes

$$\chi_s=\chi_s^{(0)}+\chi_s^{(1)} \tag{4·16}$$

$$\chi_s^{(0)}=\frac{M^{(0)}}{VH}=\frac{3}{2}n\frac{\mu_0^2}{\zeta_0}\left(1-\frac{\pi^2}{12}\left(\frac{\kappa T}{\zeta_0}\right)^2+\cdots\right), \tag{4·17}*$$

$$n=N/V$$

$$\chi_s^{(1)}=\frac{1}{VH}\left(\frac{\partial}{\partial H}\left[-R_0^{(H)}+\beta A_0^{(H)}+\beta A_1^{(H)}+\beta A_2^{(H)}\right]\right)_{\zeta,T}. \tag{4·18}$$

$\chi_s^{(0)}$ is the Pauli spin paramagnetic susceptibility. $\chi_s^{(1)}$ is the additional susceptibility which results from the electron-phonon interaction.

§ 5. Evaluation of $\chi_s^{(1)}$

Evaluations of $R_0^{(H)}$, $A_0^{(H)}$, $A_1^{(H)}$ and $A_2^{(H)}$ are given in App. II, and we find that orders of magnitude of contributions to $\chi_s^{(1)}$ from these quantities are as follows.

$$\begin{aligned} R_0^{(H)} &\sim \chi_s^{(0)}\cdot F(\sigma/k_0)\\ A_0^{(H)} &\sim \chi_s^{(0)}\cdot F(\sigma/k_0)\\ A_1^{(H)} &\sim \chi_s^{(0)}\cdot F(\sigma/k_0)\,(T/\theta_D)^4\\ A_2^{(H)} &\sim \chi_s^{(0)}\cdot F(\sigma/k_0)^2. \end{aligned} \tag{5·1}$$

* The shift of ζ_0 can also be neglected as in $C_V^{(0)}$.

As may be seen clearly from these estimations, the contribution from βA_2 which formed the main part in the additional specific heat is now negligible in comparison with those from $R_0^{(H)}$ and $\beta A_0^{(H)}$. Retaining only main terms, we have

$$\chi_s^{(1)} \fallingdotseq -n\frac{\mu_0^2}{\zeta_0}\left(2\nu+\frac{45}{8}\left(\frac{1}{4\nu}\right)^{1/3}\right)F\left(\frac{\sigma}{k_0}\right). \tag{5·2}$$

Then the spin paramagnetic susceptibility at 0°K becomes, in our case,

$$\chi_s=\chi_s^{(0)}\left\{1-\left(\frac{4}{3}\nu+\frac{15}{4}\left(\frac{1}{4\nu}\right)^{1/3}\right)F\left(\frac{\sigma}{k_0}\right)\right\} \tag{5·3}$$

$$\chi_s^{(0)}=3n\mu_0^2/2\zeta_0.$$

The correction term is very small owing to the factor σ/k_0 which is of the order $10^{-2}\sim10^{-3}$ (See eq. (6·4)). Numerical values will be discussed in the next section.

§ 6. Numerical results and discussions

As mentioned already, formulas (3·13) and (5·9) will show, if our perturbation procedure converges, how the specific heat and the spin paramagnetic susceptibility of conduction electrons are affected by the presence of the electron-phonon interaction. The discussion of the convergence of this perturbation expansion is so complicated that the complete treatment will rather be impossible. Here we are forced to be contented with assuming that the perturbation calculation up to terms of the second order of the interaction Hamiltonian gives a good approximation in the case of the metals for which the parameters $(4\nu)^{1/3}F$ are smaller than unity. The estimate of this parameter will be given in the following.

Bloch's theory of electrical conductivity of metals which is based upon the same electron-phonon interaction Hamiltonian as ours, shows that the resistivity ρ at room temperatures T_r is given by the formula

$$\rho=\frac{3\pi^3}{2}\frac{\hbar\kappa}{e^2}\frac{k_0F}{\nu q_m^{\,2}\zeta_0}T_r, \tag{6·1}$$

which leads to

$$(4\nu)^{1/3}F/\zeta_0=\frac{\nu\rho}{T_r}\left(\frac{N_0}{V_A}\right)^{1/3}\frac{e^2}{\hbar\kappa}\frac{2^{7/3}}{3^{2/3}\pi^{7/3}} \tag{6·2}$$

$$=2.244\frac{\nu\rho}{T_rV_A^{1/3}}\times10^{31}\ \text{c.g.s. units.} \tag{6·3}$$

The additional specific heat coefficient $\gamma^{(1)}$ in eq. (3·7) can be evaluated directly from experimental values of ρ because it is determined only by the factor $(4\nu)^{1/3}F/\zeta_0$ given in (6·3) as can be seen easily. For the determination of the additional susceptibility $\chi_s^{(1)}$, it is necessary to know the values of another factor σ/k_0, for the evaluation of which experimental values of the Debye temperature θ_D are needed. We use the relation

$$\frac{\sigma}{k_0}=\frac{1}{2}(4\nu)^{1/3}\frac{\kappa\theta_D}{\zeta_0}, \tag{6·4}$$

and hence

$$F\left(\frac{\sigma}{k_0}\right)=\frac{1}{2}(4\nu)^{1/3}\frac{F}{\zeta_0}\cdot\kappa\theta_D. \tag{6·5}$$

As is well known, ζ_0 contains the electron mass which is in our treatment essentially equal to the effective mass that represents the effects of the periodic field of the lattice. For alkali metals, the theoretical values of the above mentioned effective mass given by Brooks,[6] and for Cu, Ag and Au, those values by Kambe[7] can be used. Numerical values of various quantities discussed above are given in Tables 1 and 2.

Table 1. The upper values* of ρ are observed at 20°C and the lower values** at temperatures given on the right-hand side, and corresponding values of quantities $(4\nu)^{1/3}F\zeta_0$ in Table 1; $(4\nu)^{1/3}F$, $F(\sigma/k_0)$, in Table 2; $\gamma^{(1)}/\gamma^{(0)}$, $\chi_s^{(1)}/\chi_s^{(0)}$ in Table 3; $\gamma_{calc.}$, $\chi_{s\,calc.}$ in Table 4 are also given on each lower side.

	V_A c.c./mol	ρ 10^{-16} esu	θ_D °K	$(4\nu)^{1/3}F\zeta_0$ 10^{12}	m_{eff}/m_0
Li	13.0	0.10	350	0.325	1.45
Na	23.6	0.054	158	0.143	0.98
		0.024 150°K		0.126	
K	44.7	0.079	99.3	0.168	0.93
		0.016 80°K		0.126	
Rb	55.8	0.15	60	0.302	0.89
		0.021 50°K		0.252	
Cs	71.0	0.22	25	0.407	0.83
		0.021 35°K		0.330	
Cu	7.1	0.018	339	0.072	1.017
Ag	10.2	0.018	225	0.063	0.990
Au	10.2	0.025	165	0.088	0.994

* By J. Bardeen Phys. Rev. 80 (1950), 567.
** By Handbuch der Physik XIV (1956) p. 175.

The large difference of the magnitude between values of $\gamma^{(1)}$ $\gamma^{(0)}$ and $\chi_s^{(1)}$ $\chi_s^{(0)}$ might be seen curious at first sight, if we consider that both γ and χ_s are determined by the level density at the Fermi surface in the theoretical treatment based on the one-electron approximation. However, inspecting the course of our analysis, we can find that the modifications of γ and χ_s arise from two origins, and one of which makes a marked difference between $\gamma^{(1)}$ $\gamma^{(0)}$ and $\chi_s^{(1)}$ $\chi^{(0)}$. One of the origins is the shift of the one-electron level† due to the interaction and the other is the temperature or the magnetic field dependence of the interaction energy of the system as a whole. The former gives the effects of the order $F(\sigma\ k_0)$ in both cases. The latter origin which

† This is given by eq. (2·4).

Table 2

	ζ_0 10^{-12} erg	$(4\nu)^{1/3}F$	$F(\sigma/k_0)$	$\gamma^{(0)}$ 10^{-4} cal/mol	$\chi_s{}^{(0)}$ 10^{-6} c.g.s./mol
Li	5.196	1.677	0.00785	2.613	15.00
Na	5.166	0.726 0.637	0.00157 0.00138	2.629	19.09
K	3.558	0.582 0.436	0.00115 0.00086	3.816	21.91
Rb	3.206	0.976 0.813	0.00125 0.00104	4.236	24.32
Cs	2.927	1.191 0.965	0.00070 0.00057	4.639	26.63
Cu	11.085	0.814	0.00168	1.225	7.034
Ag	8.948	0.558	0.00098	1.518	8.71
Au	8.912	0.778	0.00100	1.524	8.75

depends on the occupation of the pairs of the one-electron levels have very different effects on γ and χ_s respectively corresponding to the differences in the modification of electron configuration by the temperature agitation and the modification coming from the influence of the magnetic field.

The usual method of estimating the effective mass of the electron in the periodic field of the lattice, from the observed value of γ by the Sommerfeld formula, should be revised by taking into account this $\gamma^{(1)}$ together with the correction term $\gamma^{(0)}\Delta_\gamma$ resulting from the Coulomb interaction to be discussed below.

As shown in the second column in Table 2, the magnitude of the parameter $(4\nu)^{1/3}F$ cannot be said to be fairly small. For Li it is particularly large. Here, however, we can say nothing about the precision of our approximation as has been mentioned in the beginning of this section. For that purpose, at least the evaluation of the fourth order term of the interaction Hamiltonian will be required.

The contribution of the additional susceptibility $\chi_s^{(1)}$ is about 1%, and this is consistent with the fact that Pines' χ_s which considered only the Coulomb interaction show excellent agreement with experimental values for Li and Na.

The effect of the Coulomb interaction on γ and χ_s has been investigated by Pines[7] by using Bohm-Pines formalism, and for γ there is another treatment by Matsudaira[8] basing on the same starting point as Pines. Here we do not reproduce their formulas in detail, but their results can be written as follows,

$$\gamma=\gamma^{(0)}(1+\Delta_\gamma) \tag{6·6}$$

$$\chi_s=\chi_s^{(0)}(1+\Delta_\chi), \tag{6·7}$$

where Δ_γ and Δ_χ take the numerical values in Table 3. There are some discrepancies between the values by Pines and those by Matsudaira, but we cannot decide here which is the appropriate one.

When the effects of the electron-phonon interaction and that of the Coulomb interac-

Table 3

	$\gamma^{(1)}/\gamma^{(0)}$	$\chi_s^{(1)}/\chi_s^{(0)}$	$\Delta_\gamma{}^{P*}$	$\Delta_\gamma{}^{M**}$	Δ_χ
Li	0.8385	−0.0290	−0.16	−0.480	0.623
Na	0.363	−0.0058	−0.10	−0.403	0.325
	0.3185	−0.0051			
K	0.291	−0.0043	−0.07	−0.399	0.259
	0.218	−0.0032			
Rb	0.488	−0.0046	−0.05	−0.393	0.182
	0.407	−0.0039			
Cs	0.595	−0.0026	−0.04	−0.376	0.132
	0.483	−0.0021			
Cu	0.407	−0.0062			
Ag	0.279	−0.0036			
Au	0.389	−0.0037			

* $\Delta_\gamma{}^P$ means the values by Pines[5] and
** $\Delta_\gamma{}^M$ means the values by Matsudaira[8]

tion are considered at the same time, there will appear some cross effects. These effects are very complicated and we cannot enter into their investigation now. Here, however, we want to use tentatively following formulas for γ and χ_s which can be obtained by merely combining these effects without considering cross terms.

$$\gamma=\gamma^{(0)}\left(1+\Delta_\gamma+\frac{1}{2}(4\nu)^{1/3}F\right), \tag{6·8}$$

$$\chi_s=\chi_s^{(0)}\left(1+\Delta_\chi-\left(\frac{4\nu}{3}+\frac{15}{4}\left(\frac{1}{4\nu}\right)^{1/3}\right)F\left(\frac{\sigma}{k_0}\right)\right). \tag{6·9}$$

The values of γ and χ_s given by these formulas are shown in Table 4. Unfortunately, the experimental values of γ and χ_s suitable for our purposes are very few and we cannot test even the qualitative tendency predicted by these formulas.

From (6·9) and (6·8) we obtain

$$\frac{\chi_s}{\gamma}=3\left(\frac{\mu_0}{\pi\kappa}\right)^2\frac{1+\Delta_\chi-(4\nu/3+15/4\cdot(1/4\nu)^{1/3})F(\sigma/k_0)}{1+\Delta_\gamma+1/2\cdot(4\nu)^{1/3}F}. \tag{6·10}$$

This formula shows the deviation of χ_s/γ from the ideal gas value $3(\mu_0/\pi\kappa)^2=1.37\times10^{-4}$ (in c.g.s. units) which is a universal constant. The values of this quantity for alkali metals are plotted in Fig. 1 together with the values by the formula

$$\frac{\chi_s}{\gamma}=3\left(\frac{\mu_0}{\pi\kappa}\right)^2\frac{1+\Delta_\chi}{1+\Delta_\gamma} \tag{6·11}$$

which contains only the effects of the Coulomb interaction. The abscissa in Fig. 1 is the values of dimensionless parameter r_s defined by the relation

$$4\pi(r_s a_0)^3/3=V_A/N_0$$

where a_0 is the Bohr radius and N_0 is the Avogadro number. Among these metals, sodium is the only one for which experimental data can be used for the test of this relation.

For Cu, Ag and Au, observed values of χ_s/γ are very large compared with the ideal gas value. There might be some possibilities that the terms Δ_γ and Δ_χ in eq. (6·10) will account for this discrepancy, though there are no available data on these terms for the above metals. However, considering the ambiguities in the determination of χ_s for these metals, we cannot place so much confidence on these observed values.

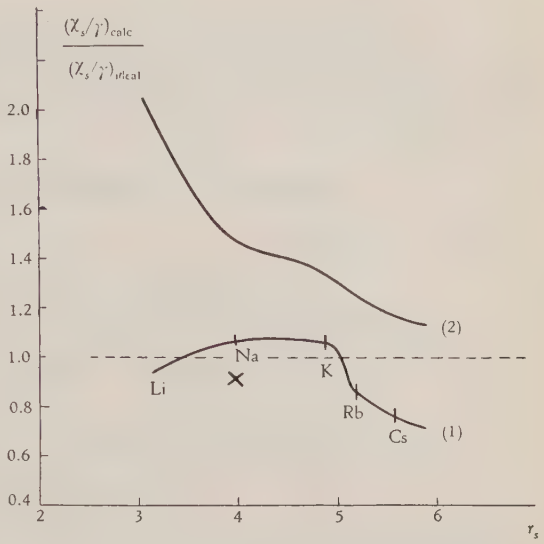

Fig. 1. (1) by eq. (6·10), (2) by eq. (6·11), × Experiment for Na

Table 4. γ and χ_s are calculated by eqs. (6·8) and (6·9), where for Δ_Γ we have adopted $\Delta_\Gamma{}^P$ in Table 3.

	$\gamma_{obs.}$ 10^{-4} cal/mol	$\gamma_{calc.}$ "	χ_s obs. 10^{-6}c.g.s./mol	χ_s calc. "	$\frac{(\chi_s/\gamma)_{obs.}}{(\chi_s/\gamma)_{ideal}}$	$\frac{(\chi_s/\gamma)_{calc.}}{(\chi_s/\gamma)_{ideal}}$	**
Li		4.386	26.37	23.907		0.949	1.932
Na	4.3	3.320	22.54	19.904	0.914	1.045	1.472
		3.202		19.915		1.084	
K		4.612		27.288		1.029	1.344
		4.381		27.312		1.086	
Rb		6.091		28.625		0.819	1.244
		5.744		28.645		0.864	
Cs		7.121		30.061		0.735	1.178
		6.596		30.075		0.794	
Cu	1.78	*1.724	22.25	6.991	2.179	*0.706	
Ag	1.60—1.64	*1.942	32.36	8.681	3.525—3.661	*0.779	
Au	1.77	*2.142	43.01	8.714	4.242	*0.717	

* The values of γ and χ_s for Cu, Ag and Au do not contain the effect of the Coulomb interaction.

** The last column shows the values of $(\chi_s/\gamma)_{calc.}/(\chi_s/\gamma)_{ideal}$ in which the theoretical values are taken from Pines' result.

Summarizing the above discussions on numerical values, we are inclined to stress, in spite of some ambiguities in the theory and the experimental data, that the influence of the electron-phonon interaction on specific heat and spin paramagnetic susceptibility of metals is not so small as can be quite neglected. To estimate the precision of our theoretical value for γ, the calculation of the fourth order terms of the interaction Hamiltonian is desired.

In conclusion the author expresses his sincere gratitude to Prof. A. Harashima for very kind encouragements throughout the research and to Dr. N. Matsudaira for the kindness to give his results before publication and also to Prof. F. Nakano of Nagoya University and to the colleagues in the institute for invaluable discussions and criticisms.

Appendix I. Evaluation of Integrals in (3·2)

(i) L_1 and L_1'

$$L_1=2\int d\boldsymbol{k}\int d\boldsymbol{k}'\,\Phi(\boldsymbol{k},\ \boldsymbol{k}')\beta^2(\varepsilon_k-\zeta)(\varepsilon_{k'}-\zeta)g_k g_{k'}$$

$$q=|\boldsymbol{q}|=|\boldsymbol{k}'-\boldsymbol{k}|\leq q_m,$$

$$\Phi(\boldsymbol{k},\ \boldsymbol{k}')=\frac{q^2}{(\boldsymbol{k}'^2-\boldsymbol{k}^2)^2-\sigma^2q^2},\quad g_k=\frac{1}{(1+\exp[\beta(\varepsilon_k-\zeta)])}\frac{1}{(1+\exp[-\beta(\varepsilon_k-\zeta)])}. \tag{AI·1}$$

The integral L_1 can be evaluated easily by making use of the δ-function like property of the function g_k. We transform the integral $\int d\boldsymbol{k}\int d\boldsymbol{k}'$ to $\int dx\int dy\int dz$ by the following relations

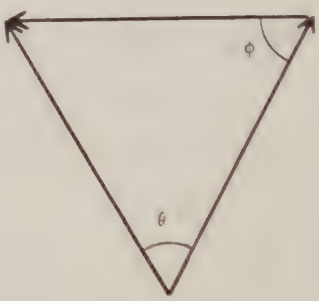

Fig. 2

$$q^2=k'^2+k^2-2kk'z,\quad z=\cos\theta$$

$$\beta(\varepsilon_k-\zeta)=x\quad\text{or}\quad k^2=\bar{k}^2\left(1+\frac{x}{\beta\zeta}\right)$$

$$\beta(\varepsilon_{k'}-\zeta)=y\quad\text{or}\quad k'^2=\bar{k}^2\left(1+\frac{y}{\beta\zeta}\right)$$

$$\frac{\hbar^2}{2m}\bar{k}^2=\zeta. \tag{AI·2}$$

We obtain

$$L_1=-\frac{4\pi^2\bar{k}^4}{\sigma^2(\beta\zeta)^2}\int_{-\beta\zeta}^{\infty}xg(x)\,dx\int_{-\beta\zeta}^{\infty}yg(y)\,dy\int_{z_m}^{1}dz$$

$$\times kk'\left(1-\frac{(y-x)^2}{(y-x)^2-(\sigma\beta\zeta\bar{k}^2)^2(k'^2+k^2-2kk'z)}\right), \tag{AI·3}$$

where z_m corresponds to the maximum value of θ with fixed k and k';

$$q_m^2=k'^2+k^2-2kk'z_m,$$

and we have used the relation

$$\frac{q^2}{(\boldsymbol{k}'^2-\boldsymbol{k}^2)^2-\sigma^2q^2}=-\frac{1}{\sigma^2}\left(1-\frac{(\boldsymbol{k}'^2-\boldsymbol{k}^2)^2}{(\boldsymbol{k}'^2-\boldsymbol{k}^2)^2-\sigma^2q^2}\right)$$

The function $g(x)$ is

$$g(x)=\frac{1}{(1+e^{x})(1+e^{-x})}.$$

After the z-integration, we get

$$L_1=-\frac{4\pi^2\bar{k}^4}{\sigma^2(\beta\zeta)^2}\int_{-\beta\zeta}^{\infty}xg(x)\,dx\int_{-\beta\zeta}^{\infty}yg(y)\,dy$$
$$\times\left(\frac{1}{2}(q_m^2-(k'^2-k^2))-\frac{(y-x)^2\bar{k}^4}{2(\beta\zeta)^2\sigma^2}\log\left|\frac{(y-x)^2-\sigma^2(\beta\zeta)^2(k'-k)^2/\bar{k}^4}{(y-x)^2-(\Theta_D/T)^2}\right|\right) \tag{AI·4}*$$

where the relation

$$\sigma\beta\zeta q_m/\bar{k}^2=\Theta_D/T \tag{AI·5}$$

is used.

Now we make following approximations considering that the main contribution comes from the neighbourhood of $x=0$ and $y=0$ owing to the presence of $g(x)$ and $g(y)$. We replace $-\beta\zeta$ with $-\infty$, and expand k, k' and logarithmic function in the integrand in powers of $x/\beta\zeta$, $y/\beta\zeta$ or $x/\beta\kappa\Theta_D$, $y/\beta\kappa\Theta_D$;

$$k'-k=\bar{k}\left(\frac{1}{2}\frac{1}{\beta\zeta}(y-x)-\frac{1}{8}\left(\frac{1}{\beta\zeta}\right)^2(y-x)^2+\frac{1}{16}\left(\frac{1}{\beta\zeta}\right)^3(y-x)^3-\cdots\right)$$
$$\log|(y-x)^2-\sigma^2(\beta\zeta)^2(k'-k)^2/\bar{k}^4|$$
$$=2\log(y-x)-\frac{1}{4}\left(\frac{\sigma}{\bar{k}}\right)^2\left(1-\frac{1}{4}\frac{1}{\beta\zeta}(y+x)+\cdots\right)^2$$
$$\log|(y-x)^2-(\Theta_D/T)^2|$$
$$=\log(\Theta_D/T)^2-(T/\Theta_D)^2(y-x)^2+\cdots.$$

Retaining only the first terms of these expansions, we get

$$L_1\fallingdotseq-\frac{4\pi^2\bar{k}^6}{(\beta\zeta)^2\sigma^2}\left\{-\frac{1}{8(\beta\zeta)^2}\iint(y-x)^2xyg(x)g(y)\,dx\,dy\right.$$
$$-\left(\frac{\bar{k}}{\sigma}\right)^2\left(\frac{1}{\beta\zeta}\right)^2\iint(y-x)^2\log(y-x)\cdot xyg(x)g(y)\,dx\,dy$$
$$\left.+\frac{1}{2}\left(\frac{\bar{k}}{\sigma}\right)^2\left(\frac{1}{\beta\zeta}\right)^2\log\left(\frac{\Theta_D}{T}\right)\iint(y-x)^2xyg(x)g(y)\,dx\,dy\right\}. \tag{AI·6}$$

Making use of the formulas

* In this integral and in similar ones which will appear afterwards, we use the principal value evaluation where the denominator in the integrand becomes zero.

$$\int_{-\infty}^{\infty} x^{2l+1} g(x)\,dx = 0 \qquad \int_{-\infty}^{\infty} x^2 g(x)\,dx = \frac{\pi^2}{3}$$

$(l = 0,\ 1,\ 2, \cdots)$

and the approximation

$$\int_{-\infty}^{\infty}\int xy(y-x)^2 \log(y-x)\, g(x)\, g(y)\,dx\,dy = 0,$$

we have

$$L_1 \fallingdotseq -\frac{4\pi^2 \bar{k}^6}{\sigma^2(\beta\zeta)^2}\left\{\frac{\pi^4}{36}\left(\frac{1}{\beta\zeta}\right)^2 + \frac{2\pi^4}{9}\left(\frac{\bar{k}}{\sigma}\right)^2\left(\frac{1}{\beta\zeta}\right)^2 \log\left(\frac{\Theta_D}{T}\right)^2\right\}$$

$$= -\frac{4\pi^2 k_0^6}{\sigma^2(\beta\zeta_0)^2}\left\{\frac{\pi^4}{36}\left(\frac{1}{\beta\zeta_0}\right)^2\left(\frac{k_0}{\bar{k}}\right)^2 + \frac{2\pi^4}{9}\left(\frac{k_0}{\sigma}\right)^2\left(\frac{1}{\beta\zeta_0}\right)^2 \log\left(\frac{\Theta_D}{T}\right)^2\right\}. \qquad \text{(AI·7)}$$

The first term can be neglected compared with the second. Then the final result is

$$\frac{8\nu F\sigma^2\zeta_0\Omega^2}{3NT} L_1 \fallingdotseq N\frac{2\pi^4}{3}(4\nu)^{1/3} F \frac{\kappa^4 T^3}{\zeta_0(\kappa\Theta_D)^2} \log\left(\frac{\Theta_D}{T}\right). \qquad \text{(AI·8)}$$

The evaluation of L_1' can be done in a similar manner and we find that this gives no contribution of the order under consideration due to the appearance of the factor $xyg(x) \times g(y)$ in place of $xyg(x)g(y)$ in L_1.

(ii) L_2 and L_2'

$$L_2 = 2\int d\boldsymbol{k}\int d\boldsymbol{k}'\, \Phi(\boldsymbol{k},\ \boldsymbol{k}')\beta^2(\varepsilon_k - \zeta)^2 h_k f_{k'}$$

$$q = |\boldsymbol{k}' - \boldsymbol{k}| \leq q_m. \qquad \text{(AI·9)}$$

The evaluation of this integral is considerably tedious. We substitute f_k with its 0 K form $f_k^{(0)}$ neglecting the temperature dependent term which comes from this part. Transforming $\int d\boldsymbol{k}\int d\boldsymbol{k}'$ back to $\int d\boldsymbol{k}\int d\boldsymbol{q}$, we see that the presence of $f_k^{(0)}$ is equivalent to the following correlations among the regions of the variables k, q, and ϑ (Fig. 2). We have

$$L_2 = 16\pi^2 \int_0^{\infty} k^2\,dk \int_0^{q_m} q^2\,dq \int_{-1}^{1} dz\, \Phi(\boldsymbol{k},\ \boldsymbol{k}')\beta^2(\varepsilon_k - \zeta)^2 h_k f_{k'}^{(0)}$$

$$= 16\pi^2\left\{\left[\int_0^{\bar{k}-q_m} k^2\,dk \int_0^{q_m} q^2\,dq \int_{-1}^{1} dz + \int_{\bar{k}-q_m}^{\bar{k}} \int_0^{\bar{k}-k} \int_{-1}^{1} + \int_{\bar{k}-q_m}^{\bar{k}} \int_{\bar{k}-k}^{q_m} \int_{z_c}^{1} + \int_{\bar{k}}^{\bar{k}+q_m} \int_{k-\bar{k}}^{q_m} \int_{z_c}^{1}\right]\right.$$

$$\left.\times \Phi(\boldsymbol{k},\ \boldsymbol{k}')\beta^2(\varepsilon_k - \zeta)^2 h_k\right\} \qquad (\bar{k} > q_m)$$

$$=16\pi^2\left\{\left[\int_0^{\bar{k}}\int_0^{\bar{k}-k}\int_{-1}^{1}+\int_0^{\bar{k}}\int_{\bar{k}-k}^{q_m}\int_{z_c}^{1}+\int_{\bar{k}}^{\bar{k}+q_m}\int_{k-\bar{k}}^{q_m}\int_{z_c}^{1}\right]\right.$$

$$\left.\times\Phi(\boldsymbol{k},\ \boldsymbol{k}')\,\beta^2(\varepsilon_k-\zeta)^2h_k\right\}\quad(\bar{k}<q_m)\qquad\text{(AI·10)}$$

where z_c is given by

$$\bar{k}^2=q^2+k^2-2kqz_c,$$

and

$$\Phi(\boldsymbol{k},\boldsymbol{k}')=\frac{1}{2\sigma}\left(\frac{1}{q-2kz-\sigma}-\frac{1}{q-2kz+\sigma}\right).$$

The function h_k is very small except in the neighbourhood of $k=\bar{k}$. Hence the first term in the case $\bar{k}>q_m$ can be neglected and the difference of the lower limit of the k-integral between both cases becomes immaterial. After the z-integration and subsequent rearrangements, we get

$$L_2=\frac{4\pi^2}{\sigma}(J^{(1)}+J^{(2)}+J^{(3)})$$

$$J^{(1)}=\int_0^{\bar{k}}\beta^2(\varepsilon_k-\zeta)^2h_k\,kdk\int_0^{\bar{k}-k}q^2dq\left(\log\left|\frac{q-2k+\sigma}{q-2k-\sigma}\right|+\log\left|\frac{q+2k-\sigma}{q+2k+\sigma}\right|\right)$$

$$J^{(2)}=\int_0^{\bar{k}}\beta^2(\varepsilon_k-\zeta)^2h_k\,kdk\int_{\bar{k}-k}^{q_m}q^2dq\left(\log\left|\frac{q-2k+\sigma}{q-2k-\sigma}\right|+\log\left|\frac{\bar{k}^2-k^2-\sigma q}{\bar{k}^2-k^2+\sigma q}\right|\right)$$

$$J^{(3)}=\int_{\bar{k}}^{\bar{k}+q_m}\beta^2(\varepsilon_k-\zeta)^2h_k\,kdk\int_{k-\bar{k}}^{q_m}q^2dq\left(\log\left|\frac{q-2k+\sigma}{q-2k-\sigma}\right|+\log\left|\frac{\bar{k}^2-k^2-\sigma q}{\bar{k}^2-k^2+\sigma q}\right|\right).$$

(AI·11)

The q-integration can be carried out easily as follows.

$$\int q^2\log|q+A|dq=\frac{1}{3}q^3\log|q+A|$$

$$-\frac{1}{3}\left\{\frac{1}{3}q^3-\frac{1}{2}q^2A+qA^2-A^3\log|q+A|\right\}.\qquad\text{(AI·12)}$$

Then we change the variable k to x as in eq. (AI·2), and utilize the above mentioned behaviour of $h(x)$. Considering the expansion of k in powers of $x/\beta\zeta$, we get

$$J^{(1)}=\frac{\bar{k}^2}{2\beta\zeta}\int_{-\infty}^{\infty}x^2h(x)\,dx\left(\text{terms of the order }\left(\frac{x}{\beta\zeta}\right)^3\right) \tag{AI·13}$$

and the contribution to $C_V^{(1)}$ from this part turns out to be of the order

$$\frac{\kappa^4T^3}{\zeta_0^3}F\left(\frac{\sigma}{k_0}\right)^2.$$

Now we combine the first terms of $J^{(2)}$ and $J^{(3)}$, and get

$$\begin{aligned}&\frac{\bar{k}^2}{2\beta\zeta}(2\sigma)\int_{-\infty}^{\infty}x^2h(x)\,dx\left\{\left[\frac{1}{2}q_m^2+2\bar{k}q_m+4\bar{k}^2\log\frac{q_m-2\bar{k}}{2\bar{k}}\right]\right.\\&\quad+\left(\frac{x}{\beta\zeta}\right)\left[\bar{k}q_m-2\bar{k}^2+4\bar{k}^2\log\frac{2\bar{k}-q_m}{2\bar{k}}+4\bar{k}^2\frac{\bar{k}}{2\bar{k}-q_m}\right]\\&\quad\left.+O\left(\left(\frac{x}{\beta\zeta}\right)^2\right)\right\}+O(\sigma^2).\end{aligned} \tag{AI·14}$$

The contribution from this first bracket vanishes because of the presence of $h(x)$ and from the next we obtain a contribution of the order

$$\frac{\kappa^2T}{\zeta_0}F\left(\frac{\sigma}{k_0}\right)^2$$

to $C_V^{(1)}$.

Combination of the second terms of $J^{(2)}$ and $J^{(3)}$ yields

$$\begin{aligned}&\frac{\bar{k}^2}{2\beta\zeta}\left[\int_{-\infty}^{0}x^2h(x)\,dx\int_{\bar{k}-q_m}^{q_m}q^2dq+\int_{0}^{\infty}x^2h(x)\int_{k-\bar{k}}^{q_m}q^2dq\right]\times\log\frac{q+(\bar{k}^2x/\sigma_l\beta\zeta)}{q-(\bar{k}^2x/\sigma_l\beta\zeta)}\\&=\frac{\bar{k}^2}{2\beta\zeta}\int_{-\infty}^{\infty}x^2h(x)\,dx\left\{\frac{1}{3}\left(q_m^3\log\left|\frac{q_m+(\bar{k}^2x/\sigma_l\beta\zeta)}{q_m-(\bar{k}^2x/\sigma_l\beta\zeta)}\right|\right.\right.\\&\quad+\frac{\bar{k}^2x}{\sigma_l\beta\zeta}q_m^2+\left(\frac{\bar{k}^2x}{\sigma_l\beta\zeta}\right)^3\log q_m^2-\left.\left(\frac{\bar{k}^2x}{\sigma_l\beta\zeta}\right)^2\right)\\&\quad-\frac{1}{3}\left((\bar{k}-k)^3\log\left|\frac{\bar{k}-k+(\bar{k}^2x/\sigma_l\beta\zeta)}{\bar{k}-k-(\bar{k}^2x/\sigma_l\beta\zeta)}\right|\right.\\&\quad\left.\left.+\frac{\bar{k}^2x}{\sigma_l\beta\zeta}(\bar{k}-k)^2+\left(\frac{\bar{k}^2x}{\sigma_l\beta\zeta}\right)^3\log(\bar{k}-k)^2-\left(\frac{\bar{k}^2x}{\sigma_l\beta\zeta}\right)^2\right)\right\}\\&=\frac{\bar{k}^4q_m^2}{2\sigma(\beta\zeta)^2}\left\{\int_{-\infty}^{\infty}x^3h(x)\,dx\right.\end{aligned}$$

$$+\left(\frac{2}{9}\left(\frac{T}{\theta_D}\right)^2-\frac{2}{3}\left(\frac{T}{\theta_D}\right)^2\log\left(\frac{T}{\theta_D}\right)\right)\int_{-\infty}^{\infty}x^5h(x)\,dx$$

$$-\frac{2}{3}\left(\frac{T}{\theta_D}\right)^2\int_{-\infty}^{\infty}x^5\log x\,h(x)\,dx\Big\}+\cdots \qquad \text{(AI·15)}$$

and we find that this part gives the largest contributions to $C_V^{(1)}$ among various terms in L_2. The values of the integrals in this expression are given as follows.

$$\int_{-\infty}^{\infty}x^3h(x)\,dx=\int_{-\infty}^{\infty}\frac{x^3(e^x-1)}{(1+e^x)^3}dx=\pi^2$$

$$\int_{-\infty}^{\infty}x^5h(x)\,dx=\frac{7}{3}\pi^4 \qquad \text{(AI·16)}$$

$$\int_{-\infty}^{\infty}x^5\log x\,h(x)\,dx=\frac{7}{15}\pi^4+l_4$$

$$l_4=\int_{-\infty}^{\infty}x^4\log x\,g(x)\,dx.$$

The numerical values of l_4 can be obtained by numerical integration if necessary.

Retaining only these largest terms, we finally get

$$L_2\fallingdotseq\frac{2\pi^2q_m{}^2\bar{k}^4}{\sigma^2(\beta\zeta)^2}\left\{\pi^2+\left(\frac{T}{\theta_D}\right)^2\left[\frac{28}{135}\pi^4-\frac{10}{3}l_4+\frac{14}{9}\pi^4\log\left(\frac{\theta_D}{T}\right)\right]\right\} \qquad \text{(AI·17)}$$

and

$$\frac{8\nu F\sigma^2\zeta_0\Omega^2}{3NT}L_2\fallingdotseq N\frac{\pi^2\kappa^2T}{2\zeta_0}\cdot\frac{3}{2}(4\nu)^{1/3}F$$

$$+N\frac{\kappa^4T^4}{\zeta_0(\kappa\theta_D)^2}\frac{3}{4}(4\nu)^{1/3}F\left[\frac{28}{135}\pi^4+\frac{14}{9}\pi^4\log\left(\frac{\theta_D}{T}\right)-\frac{10}{3}l_4\right]. \qquad \text{(AI·17′)}$$

The evaluation of L_2' goes in a quite similar manner and the expression corresponding to (AI·13), (AI·14) and (AI·15) are obtained. The only difference found there is the appearance of $xh(x)$ instead of $h(x)$. Hence we easily see that the parts of L_2' which correspond to (AI·13) and (AI·15) vanish approximately and the one which corresponds to (AI·14) gives a contribution of the order $\kappa^2TF(\sigma/k_0)^2/\zeta_0$ to $C_V^{(1)}$.

(iii) L_3 and L_3'

$$L_3=-4\int d\boldsymbol{k}\int d\boldsymbol{k}'\,\Phi(\boldsymbol{k},\,\boldsymbol{k}')\,\beta(\varepsilon_k-\zeta)\,g_kf_{k'}. \qquad \text{(AI·18)}$$

The evaluation of this integral can be done also in a similar manner as the evaluation of L_2. The parts of this L_3 which corresponds to (AI·13), (AI·14) and (AI·15) of

L_2 give corresponding contributions. Hence it is sufficient for us to write down here the largest one which corresponds to (AI·15).

$$\frac{q^2\bar{k}^4}{2\sigma(\beta\zeta)^2}\Big\{\int_{-\infty}^{\infty}x^2 g(x)\,dx + \Big(\frac{2}{9}\Big(\frac{T}{\theta_D}\Big)^2-\frac{2}{3}\Big(\frac{T}{\theta_D}\Big)^2\log\Big(\frac{T}{\theta_D}\Big)\Big)\int_{-\infty}^{\infty}x^4 g(x)\,dx - \frac{2}{3}\Big(\frac{T}{\theta_D}\Big)^2\int_{-\infty}^{\infty}x^4\log x\, g(x)\,dx\Big\}. \tag{AI·19}$$

Then we find

$$\frac{8\nu F\sigma^2\zeta_0\Omega^2}{3NT}L_3 \fallingdotseq -N\frac{\pi^2\kappa^2 T}{2\zeta_0}(4\nu)^{1/3}F - N\frac{\kappa^4T^3}{\zeta_0(\kappa\theta_D)^2}\frac{3}{4}(4\nu)^{1/3}F\Big[\frac{28}{135}\pi^4+\frac{28}{45}\pi^4\log\Big(\frac{\theta_D}{T}\Big)-\frac{4}{3}l_4\Big]. \tag{AI·20}$$

We can also show that the integral L_3' gives a contribution of the order $\kappa^2TF(\sigma\bar{k})^4\zeta_1$ to $C_V^{(1)}$.

Combining the above results we obtain

$$\frac{8\nu F\sigma^2\zeta_0\Omega^2}{3N}\Big[\frac{1}{T}(L_1+L_2+L_3)+\frac{\partial\zeta}{\partial T}(L_1'+L_2'+L_3')\Big] = N\frac{\pi^2\kappa^2T}{2\zeta_0}\frac{1}{2}(4\nu)^{1/3}F + N\frac{\kappa^4T^3}{\zeta_0(\kappa\theta_D)^2}\frac{3}{4}(4\nu)^{1/3}F\Big[\frac{42}{45}\pi^4\log\Big(\frac{\theta_D}{T}\Big)-2l_4\Big].$$

Appendix II. Evaluations of $R_0^{(H)}$, $\beta A_0^{(H)}$, $\beta A_1^{(H)}$ and $\beta A_2^{(H)}$

(i) $R_0^{(H)}$

The evaluation of $R_0^{(H)}$ is very easy. We immediately obtain

$$R_0^{(H)} = \alpha\Big(\frac{\mu_0 H}{\zeta}\Big)^2(\beta\zeta)^2\Omega\frac{\hbar^2}{2m}\int k^2 b_k\,dk \fallingdotseq \alpha\Big(\frac{\mu_0 H}{\zeta}\Big)^2(\beta\zeta)^2\Omega\frac{\hbar^2}{2m}\frac{2\pi k^5}{\beta\zeta}\int_{-\infty}^{\infty}\Big(1+\frac{x}{\beta\zeta}\Big)^{3/2}h(x)\,dx = \alpha\Big(\frac{\mu_0 H}{\zeta}\Big)^2 2\pi\Omega\bar{k}^3\zeta\Big(\frac{3}{2}-\frac{3}{16}\Big(\frac{1}{\beta\zeta}\Big)^2-\cdots\Big) \fallingdotseq \alpha\Big(\frac{\mu_0 H}{\zeta_0}\Big)^2\frac{9}{8}N\zeta_0\Big(\frac{\zeta}{\zeta_0}\Big)^{1/2}. \tag{AII·1}$$

Substituting the value of α in [I]—(4·6),

$$\alpha \doteqdot \frac{5}{2} F\left(\frac{\sigma}{k_0}\right)\left(\frac{1}{4\nu}\right)^{1/3},$$

we have

$$R_0^{(H)} \doteqdot \left(\frac{\mu_0 H}{\zeta_0}\right)^2 \frac{45}{16} N\zeta_0 F\left(\frac{\sigma}{k_0}\right)\left(\frac{1}{4\nu}\right)^{1/3}. \tag{AII·2}$$

(ii) $\beta A_0^{(H)}$

$$\beta A_0^{(H)} = \left(\frac{\mu_0 H}{\zeta}\right)^2 (\beta\zeta)^2 G\Omega^2 \int d\boldsymbol{k} \int d\boldsymbol{q} \frac{q}{\boldsymbol{k}'^2 - \boldsymbol{k}^2 + \sigma q} h_k, \tag{AII·3}$$

$$q = |\boldsymbol{k}' - \boldsymbol{k}| \leq q_m$$

$$G = 4\nu F \sigma \zeta_0 / 3N.$$

Taking the integration variables shown in Fig. 2, we get

$$\beta A_0^{(H)} = \left(\frac{\mu_0 H}{\zeta}\right)^2 (\beta\zeta)^2 G\Omega^2 8\pi^2 \int_0^\infty k^2 h_k dk \int_0^{q_m} q^2 dq \int_{-1}^{1} dz \frac{1}{q+\sigma-2kz}$$

$$= \left(\frac{\mu_0 H}{\zeta}\right)^2 (\beta\zeta)^2 G\Omega^2 4\pi^2 \int_0^\infty k h_k dk \int_0^{q_m} q^2 dq \log\left|\frac{q+\sigma+2k}{q+\sigma-2k}\right|$$

$$= \left(\frac{\mu_0 H}{\zeta}\right)^2 (\beta\zeta)^2 G\Omega^2 4\pi^2 \int_0^\infty k h_k dk$$

$$\times \left\{ \frac{1}{3} q_m{}^3 \log\left|\frac{q_m+2k+\sigma}{q_m-2k+\sigma}\right| - \frac{2}{3} q_m{}^2 k - \frac{8}{3}\sigma q_m k \right.$$

$$+ \frac{1}{3}(2k+\sigma)^3 \log\left|\frac{2k+q_m+\sigma}{2k+\sigma}\right|$$

$$\left. + \frac{1}{3}(2k-\sigma)^3 \log\left|\frac{2k-q_m-\sigma}{2k-\sigma}\right| \right\}.$$

Here we make the same transformation of the variable k to x as in (AI. (ii)) and use the characteristic behaviour of $h(x)$ at $x=0$. Then we obtain

$$\beta A_0^{(H)} = \left(\frac{\mu_0 H}{\zeta}\right)^2 G(\beta\zeta)^2 \Omega^2 4\pi^2 \frac{\bar{k}^3}{2(\beta\zeta)^2} \left\{ -\frac{2}{3} \frac{q_m{}^4}{(2\bar{k})^2 - q_m{}^2} - \frac{1}{3} q_m{}^2 \right.$$

$$\left. + 4\bar{k}^2 \log\left|\frac{(2\bar{k})^2 - q_m{}^2}{(2\bar{k})^2}\right| + \frac{2}{3} \frac{(2\bar{k})^4}{(2\bar{k})^2 - q_m{}^2} + O\left(\left(\frac{1}{\beta\zeta}\right)^2, \left(\frac{\sigma}{k_0}\right)\right) \right\} \tag{AII·4}$$

and hence

$$\beta A_0^{(H)} \doteqdot -\left(\frac{\mu_0 H}{\zeta_0}\right)^2 \nu F\left(\frac{\sigma}{k_0}\right) \zeta_0 L(\gamma) \tag{AII·5}$$

$$L(\gamma)=\frac{1}{1-\gamma^2}+\frac{1}{2}\frac{(1+\gamma^2)\gamma^2}{1-\gamma^2}-\frac{1}{2}\log|1-\gamma^2|,\quad \gamma=q_m/2k_0.$$

(iii) $\beta A_1^{(H)}$

$$\beta A_1^{(H)}=\left(\frac{\mu_0 H}{\zeta}\right)^2(\beta\zeta)^2 G\Omega^2\int dk\int dq\left(\frac{q}{e^{\beta\hbar\omega q}-1}\right)\left(-\frac{2(k'^2-k^2)}{(k'^2-k^2)^2-\sigma^2q^2}\right)h_k \qquad \text{(AII·6)}$$

$$q=|k'-k|\leq q_m.$$

We can proceed as in the appendix of the paper [I] and find

$$\beta A_1^{(H)}\sim NF\zeta_0\left(\frac{\sigma}{k_0}\right)\left(\frac{\kappa T}{\kappa\Theta_D}\right)^4. \qquad \text{(AII·7)}$$

(iv) $\beta A_2^{(H)}$

$$\beta A_2^{(H)}=\left(\frac{\mu_0 H}{\zeta}\right)^2 2\sigma G(\beta\zeta)^2\Omega^2(K_1+K_2)$$

$$K_1=\int dk\int dq\frac{q^2}{(k'^2-k^2)^2-\sigma^2q^2}g_k g_{k'} \qquad \text{(AII·8)}$$
$$q=|k'-k|\leq q_m$$

$$K_2=\int dk\int dq\frac{q^2}{(k'^2-k^2)^2-\sigma^2q^2}f_{k'}h_k.$$
$$q=|k'-k|\leq q_m$$

Evaluations of K_1 and K_2 can be carried out in a similar manner as those of L_2' in App. I. So we do not give the details of calculations here. We find that this $\beta A_1^{(H)}$ contributes to the free energy terms of the order $\zeta_0 F(\sigma/k_0)^2(\mu_0 H/\zeta)^2$. The terms in K_1 and K_2 which should give rise to large contributions corresponding to those of L_2 and L_3 cancel completely in this case.

Appendix III. Evaluation of the Integral $I_2^{(0)}$

The evaluation of $I_2^{(0)}$ is repeated here, because the evaluation given in [I]—App. IV has been done with too simplified approximations.

$$I_2^{(0)}=\Omega^2\int dk\int dk'\frac{q^2 f_k^{(0)}f_{k'}^{(0)}}{(k'^2-k^2)^2-\sigma^2q^2}$$

$$=\Omega^2\int dk\int dk'\frac{f_k^{(0)}f_{k'}^{(0)}}{(q-2kz)^2-\sigma^2} \qquad \text{(AIII·1)}$$

$$q=|k'-k|\leq q_m$$

$$k'^2=k^2+q^2-2kqz,\quad z=\cos\theta.$$

The condition $q\leq q_m$ imposed on q and the characteristic behaviour of the functions $f_k^{(0)}$ lead to somewhat complicated correlations among the regions of the integration variables k and q, hence the integral can be divided into three parts as in App. I.

$$I_2^{(0)}=16\pi^2\Omega^2\Big[\int_0^{\bar{k}-q_m}k^2dk\int_0^{q_m}q^2dq\int_{-1}^{1}dz+\int_{\bar{k}-q_m}^{\bar{k}}k^2dk\int_0^{\bar{k}-k}q^2dq\int_{-1}^{1}dz$$

$$+\int_{\bar{k}-q_m}^{\bar{k}}k^2dk\int_{\bar{k}-k}^{q_m}q^2dq\int_{z_c}^{1}dz\Big]\times\Phi(k,\ q,\ z) \qquad \text{(AIII}\cdot 2\text{)}$$

$$(q_m<\bar{k})$$

$$\Phi(k,\ q,\ z)=\frac{1}{2\sigma}\Big(\frac{1}{q-2kz+\sigma}-\frac{1}{q-2kz+\sigma}\Big)$$

and

$$I_2^{(0)}=16\pi^2\Omega^2\Big[\int_0^{\bar{k}}k^2dk\int_0^{\bar{k}-k}q^2dq\int_{-1}^{1}dz+\int_0^{\bar{k}}k^2dk\int_{\bar{k}-k}^{q_m}q^2dq\int_{z_0}^{1}dz\Big]\times\Phi(k,\ q,\ z) \qquad \text{(AIII}\cdot 2'\text{)}$$

$$(q_m>\bar{k}),$$

where $q_m\lesseqgtr k$ is essentially equivalent to the condition $q_m\lesseqgtr k_0$. Considering the definition of k and q, we find that this corresponds to the condition that the number of the conduction electrons per atom is larger, or smaller, than 2, that is

$$q_m\lesseqgtr k_0 \longleftrightarrow \nu\gtreqless 2.$$

The integration limit is given by the relation given in (AI·10).

The z-integration and the subsequent rearrangements lead to

$$I_2^{(0)}=\frac{4\pi^2\Omega^2}{\sigma}\Big\{\int_0^{\bar{k}}kdk\int_0^{q_m}q^2dq\log\Big|\frac{(q-2k+\sigma)\,(q+2k-\sigma)}{(q-2k-\sigma)\,(q+2k+\sigma)}\Big|$$

$$+\int_{\bar{k}-q_m}^{\bar{k}}kdk\int_{\bar{k}-k}^{q_m}q^2dq\log\Big|\frac{(q+2k+\sigma)\,(\bar{k}^2-k^2-\sigma q)}{(q+2k-\sigma)\,(\bar{k}^2-k^2+\sigma q)}\Big|\Big\} \qquad \text{(AIII}\cdot 4\text{)}$$

$$(q_m<\bar{k})$$

and

$$I_2^{(0)}=\frac{4\pi^2\Omega^2}{\sigma}\Big\{\int_0^{\bar{k}}kdk\int_0^{q_m}q^2dq\log\Big|\frac{(q-2k+\sigma)\,(q+2k-\sigma)}{(q-2k-\sigma)\,(q+2k+\sigma)}\Big|$$

$$+\int_0^{\bar{k}}kdk\int_{\bar{k}-k}^{q_m}q^2dq\log\Big|\frac{(q+2k+\sigma)\,(\bar{k}^2-k^2-\sigma q)}{(q+2k-\sigma)\,(\bar{k}^2-k^2+\sigma q)}\Big|\Big\} \qquad \text{(AIII}\cdot 4'\text{)}$$

$$(q_m>\bar{k}).$$

The evaluation of these double integrals to terms of the order in σ^1 is sufficient for our purposes. After lengthy but straightforward calculations, we finally get

$$I_2^{(0)} = -16\pi^2\Omega^2 Q(\bar{\gamma}) \tag{AIII·5}$$

$$Q(\bar{\gamma}) = \frac{4}{3}\bar{\gamma}^3 - \bar{\gamma}^4 \log(\sigma/2\bar{k}) + \bar{\gamma}^4 \log(1-\bar{\gamma}) + \left(\log(1-\bar{\gamma}) + \bar{\gamma} + \frac{1}{2}\bar{\gamma}^2 + \frac{1}{3}\bar{\gamma}^3 + \frac{1}{4}\bar{\gamma}^4\right) \tag{AIII·6}$$

$$(q_m < \bar{k})$$

$$Q(\bar{\gamma}) = \frac{17}{48} - \frac{1}{2}\log 2 + \frac{2}{3}\bar{\gamma}^3 + \frac{5}{8}\bar{\gamma}^4 - \bar{\gamma}^4 \log(\sigma/2\bar{k}) - \frac{1}{2}\bar{\gamma}^4 \log\bar{\gamma} + \frac{3}{2}\bar{\gamma}^4 \log(1-\bar{\gamma}) - \frac{3}{2}\left(\log(1-\bar{\gamma}) + \bar{\gamma} + \frac{1}{2}\bar{\gamma}^2 + \frac{1}{3}\bar{\gamma}^3 + \frac{1}{4}\bar{\gamma}^4\right) \tag{AIII·6'}$$

$$(q_m > \bar{k})$$

$$\bar{\gamma} = q/2\bar{k} = (1 - 4\nu)^{1/3}(k_s/\bar{k}).$$

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Relativistic Wave Equations with Maximum Spin Two

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The incompleteness in the preceding article[2] which aimed at extracting tensor equations from the canonical form of relativistic wave equations with maximum spin 2, is all removed first by completing the algebraic treatment relying upon the theory of the rotation group in five dimensions and then by performing a full analytical reduction into constituent simple fields corresponding to definite values of spin and mass.

§ 1. Introduction

The object of our investigation is the canonical form of relativistic wave equations

$$(\partial^{\rho}\alpha_{\rho}-\kappa)\psi=0 \tag{1}$$

with the additional condition that each matrix α_{ρ} is a direct sum of a certain number of independent Dirac operators γ_{ρ} specified by the commutation relations

$$\gamma_{\mu}\gamma_{\nu}+\gamma_{\nu}\gamma_{\mu}=2\delta_{\mu\nu}E. \tag{2}$$

If α_{ρ} in (1) is the γ_{ρ} itself, the wave function ψ transforms according to the representation $R_5(1/2, 1/2)$ of the rotation group in five dimensions.[1] Next, if we go on to the Duffin-Kemmer equation supposing α_{ρ} to be

$$\beta_{\rho}=(1/2)\,(\gamma_{\rho}\times E+E\times\gamma'_{\rho}) \tag{3}$$

with two independent Dirac operators γ_{ρ} and γ'_{ρ}, the wave function ψ transforms according to the representation

$$\begin{aligned} &R_5(1/2,\ 1/2)\times R_5(1/2,\ 1/2)\\ &=R_5(1,\ 1)+R_5(1,\ 0)+R_5(0,\ 0). \end{aligned} \tag{4}$$

As is well known, every irreducible representation of the rotation group in five dimensions provides a representation, usually a reducible one, in four dimensions. In our case we have

$$\begin{aligned} &R_5(0,\ 0)\to R_4(0,\ 0),\\ &R_5(1,\ 0)\to R_4(1,\ 0)+R_4(0,\ 0),\\ &R_5(1,\ 1)\to R_4(1,\ 1)+R_4(1,\ 0). \end{aligned} \tag{5}$$

According to the general formula

$$d_5(\lambda_1, \lambda_2) = (2/3)\,(\lambda_1+3/2)\,(\lambda_2+1/2)\,(\lambda_1-\lambda_2+1)\,(\lambda_1+\lambda_2+2), \tag{6}$$

$$d_4(\lambda_1, \lambda_2) = 2(\lambda_1-\lambda_2+1)\,(\lambda_1+\lambda_2+1) \text{ provided } \lambda_2 \neq 0,$$

and $$= (\lambda_1+1)^2 \text{ if } \lambda_2 = 0 \tag{7}$$

for the degrees of the representations $R_5(\lambda_1, \lambda_2)$ and $R_4(\lambda_1, \lambda_2)$, the decompositions (5) are, as to the degree, $5=4+1$ and $10=6+4$, so that they correspond to the scalar and vector theories in the Duffin-Kemmer equations.

Now we proceed to consider the theory of maximum spin 2 under the assumption that α_μ is given by

$$\alpha_\mu = {}_1\beta_\mu \times E + E \times {}_1\beta'_\mu \tag{8}$$

where ${}_1\beta_\mu$ and ${}_1\beta'_\mu$ belong to independent Duffin-Kemmer algebras. The wave function ψ comprising $4^4=256$ components transforms according to the following representations:

$$\begin{aligned} &R_5(1/2, 1/2) \times R_5(1/2, 1/2) \times R_5(1/2, 1/2) \times R_5(1/2, 1/2) \\ &= \{R_5(1, 1) + R_5(1, 0) + R_5(0, 0)\} \times \{R_5(1, 1) + R_5(1, 0) + R_5(0, 0)\} \\ &= \{R_5(1, 1) \times R_5(1, 1) + R_5(1, 0) \times R_5(1, 0) + R_5(0, 0)\} \\ &\quad + 2\{R_5(1, 1) \times R_5(1, 0) + R_5(1, 1) + R_5(1, 0)\}. \end{aligned} \tag{9}$$

In the following discussions the quantities transforming according to the representations in the first and the second brackets of the right-hand side will be called diagonal and off-diagonal parts respectively. Moreover, as was done in the preceding article,[2] hereafter referred to as I, we shall term the quantities corresponding to the representations

$$R_5(1, 0) \times R_5(1, 0) = R_5(2, 0) + R_5(1, 1) + R_5(0, 0) \tag{10}$$

and

$$R_5(1, 1) \times R_5(1, 1) = R_5(1, 0) \times R_5(1, 0) + R_5(2, 2) + R_5(2, 1) + R_5(1, 0) \tag{11}$$

in the diagonal part as lower and higher classes respectively. The first term in the right-hand side of (11) is the lower class separated from the higher. The direct product $R_5(1, 1) \times R_5(1, 0)$ appearing in the off-diagonal part in (9) is decomposable as

$$R_5(1, 1) \times R_5(1, 0) = R_5(2, 1) + R_5(1, 1) + R_5(1, 0). \tag{12}$$

Therefore in the product representation (9) we find, besides $R_5(1, 1)$, $R_5(1, 0)$ and $R_5(0, 0)$ already appearing in the Duffin-Kemmer theory, the following representations peculiar to the theory of maximum spin 2;

$$R_5(2, 2) \to R_4(2, 2) + R_4(2, 1) + R_4(2, 0), \tag{13·1}$$

$$R_5(2, 1) \to R_4(2, 0) + R_4(2, 1) + R_4(1, 0) + R_4(1, 1), \tag{13·2}$$

$$R_5(2, 0) \to R_4(2, 0) + R_4(1, 0) + R_4(0, 0). \tag{13·3}$$

The corresponding decompositions as to their degrees are

$$35=10+16+9, \tag{14·1}$$

$$35=9+16+4+6, \tag{14·2}$$

$$14=9+4+1. \tag{14·3}$$

Thus the 256 components of the wave function are separated into independent subsets each corresponding to an irreducible representation of five-dimensional rotation group. However in the preceding article the discussions were confined to the diagonal part in (9) alone and the off-diagonal one was not treated. Besides, the algebraic reductions into independent subsets were incomplete, hence leading to some difficulties in the results. In this paper we shall present the theory in a more satisfactory way by removing these deficiencies with the aid of the theory of rotation group. In the next section we shall first construct the 256 linearly independent elements in the product space of two independent Dirac algebras and then make their separation into independent subsets each corresponding to an irreducible representation appearing in (9). Moreover each set will be shown to be closed with respect to the commutation relation with $i\partial_\rho$. Then in § 3, using the same method as in I, we shall obtain the field equations in tensor form first by rearranging $16\times16=256$ components of ψ in the form of a square matrix Ψ and then by taking spur of Ψ multiplied by each one of the above linearly independent elements.

§ 2. Construction and algebraical analysis of the linearly independent elements

The linearly independent elements of the Dirac algebra are given by the elements

$$\gamma_A=\{E,\ \gamma_k \text{ and } \sigma_{mn}=(i/2)[\gamma_m,\ \gamma_n]\}, \tag{15}*$$

where $\gamma_5=\gamma_1\gamma_2\gamma_3\gamma_4$, so that $\gamma_m\gamma_n+\gamma_n\gamma_m=2\delta_{mn}E$ and that $\gamma_A^2=E$. Moreover, taking γ_ρ to be hermitian $\gamma_\rho^*=\gamma_\rho$ we have $\gamma_A^*=\gamma_A$. Now in the direct product space of two independent Dirac algebras we can construct an element

$$\begin{aligned}\Gamma&=(1/16)\sum_A{}' \varepsilon_A\gamma_A\times\gamma_A\\&=(1/16)\{E\times E+\gamma_m\times\gamma^m-(1/2)\sigma_{rs}\times\sigma^{rs}\}\end{aligned} \tag{16}$$

with $\varepsilon_A=-1$ for $\gamma_A=\sigma_{mn}$ and $\varepsilon_A=1$ otherwise. Making use of the equations

$$\begin{aligned}&[\gamma_k,\ \gamma_m]=-2i\sigma_{km},\\&[\gamma_k,\ \sigma_{rs}]=2i(\delta_{kr}\gamma_s-\delta_{ks}\gamma_r),\\&\{\gamma_k,\ \sigma_{rs}\}=-\epsilon_{krsmn}\sigma^{mn}\end{aligned} \tag{17}$$

and

* Throughout in this paper Greek indices run from one to four and Latin ones from one to five or from one to three. Sum over repeated indices is also understood. ϵ_{kmnrs} is five-dimensional antisymmetric tensor with $\epsilon_{12345}=1$. And $[X,\ Y]=XY-YX$, $\{X,\ Y\}=XY+YX$.

we get

$$[\gamma_k\times E-E\times\gamma_k,\ \Gamma']=0,$$
$$\{\gamma_k\times E-E\times\gamma_k,\ \Gamma'\}=0, \tag{18}$$

and accordingly

$$(\gamma_k\times E)\,\Gamma'=(E\times\gamma_k)\,\Gamma',$$
$$\Gamma'(\gamma_k\times E)=\Gamma'(E\times\gamma_k). \tag{19}$$

Hence we have

$$(\gamma_A\times E)\,\Gamma'=\varepsilon_A(E\times\gamma_A)\,\Gamma',$$
$$\Gamma'(\gamma_A\times E)=\varepsilon_A\Gamma'(E\times\gamma_A). \tag{20}$$

Since

$$Y=(1/4)\sum_A{}'\,\mathrm{sp}(\gamma_A Y)\gamma_A,$$
$$\sum_A{}'\,\gamma_A Z\gamma_A=4\,\mathrm{sp}(Z)\cdot E \tag{21}$$

are valid for arbitrary 4×4 matrices Y and Z (see Appendix of I), we obtain

$$\begin{aligned}\Gamma'(Y\times Z)\,\Gamma'&=(1/64)\sum_A\sum_B\varepsilon_B\mathrm{sp}(\gamma_A Y)\,(\gamma_B\gamma_A\times\gamma_B Z)\,\Gamma'\\&=(1/64)\sum_A\sum_B\varepsilon_A\mathrm{sp}(\gamma_A Y)\,(E\times\gamma_B Z\gamma_A\gamma_B)\,\Gamma'\\&=(1/16)\sum_A{}'\,\varepsilon_A\mathrm{sp}(\gamma_A Y)\,\mathrm{sp}(\gamma_A Z)\,\Gamma'.\end{aligned} \tag{22}$$

As a next step, let us construct with the aid of Γ' new elements

$$\Gamma'_{A,B}=(\gamma_A\times E)\,\Gamma'(\gamma_B\times E), \tag{23}$$

total number of which is $16^2=256$. From (22) it is readily shown that

$$\Gamma'_{A,B}\Gamma'_{C,D}=\delta_{BC}\Gamma'_{A,D} \tag{24}$$

hold because of $\mathrm{sp}(\gamma_A\gamma_B)=4\delta_{AB}$. Therefore sixteen $\Gamma'_{A,A}$'s are mutually normal idempotents. We also have

$$\begin{aligned}\mathrm{sp}(\Gamma'_{A,B})&=\mathrm{sp}\{(\gamma_A\times E)\,\Gamma'\cdot\Gamma'(\gamma_B\times E)\}\\&=\mathrm{sp}\{\Gamma'(\gamma_B\times E)\,(\gamma_A\times E)\,\Gamma'\}\\&=\delta_{AB}\mathrm{sp}(\Gamma')\\&=(1/16)\,\delta_{AB}\sum_A{}'\,\varepsilon_A\mathrm{sp}(\gamma_A)^2=\delta_{AB},\end{aligned} \tag{25}$$

and

$$\begin{aligned}\sum_A{}'\,\Gamma'_{A,A}&=\sum_A{}'(\gamma_A\times E)\,\Gamma'(\gamma_A\times E)\\&=(1/16)\sum_A{}'\sum_B{}'\varepsilon_B\gamma_A\gamma_B\gamma_A\times\gamma_B\\&=(1/4)\sum_B{}'\varepsilon_B\cdot\mathrm{sp}(\gamma_B)E\times\gamma_B\\&=E\times E.\end{aligned} \tag{26}$$

Because of the hermitian character of $\gamma_{A:B}$ we have $\Gamma^{+}_{A:B}=\Gamma_{B:A}$. We thus have the following nine types of elements.

γ_A \ γ_B	E	γ_k	σ_{rs}
E	Γ	$\Gamma_{:k}$	$\Gamma_{:[rs]}$
γ_j	$\Gamma_{j:}$	$\Gamma_{j:k}$	$\Gamma_{j:[rs]}$
σ_{mn}	$\Gamma_{[mn]:}$	$\Gamma_{[mn]:k}$	$\Gamma_{[mn]:[rs]}$

Now since (3) and (19) give

$$\begin{aligned}\beta_\rho\Gamma_{A:B}&=(1/2)\,([\gamma_\rho,\gamma_A]\times E)\,\Gamma(\gamma_B\times E),\\ -\Gamma_{A:B}\beta_\rho&=(1/2)\,(\gamma_A\times E)\,\Gamma([\gamma_\rho,\gamma_B]\times E),\end{aligned}\tag{27}$$

we have

$$\beta_\rho\Gamma=\Gamma\beta_\rho=0\tag{28}$$

for $\Gamma=\Gamma_{A:B}$ with $\gamma_A=\gamma_B=E$ because of $[\gamma_\rho, E]=0$, and moreover the commutation relation

$$\begin{aligned}[\beta_\rho,\Gamma^{(\varepsilon)}_{AB}]=(1/2)\,\{&([\gamma_\rho,\gamma_A]\times E)\,\Gamma(\gamma_B\times E)\\ &+\varepsilon(\gamma_B\times E)\,\Gamma([\gamma_\rho,\gamma_A]\times E)\}\\ &+(1/2)\,\{(\gamma_A\times E)\,\Gamma([\gamma_\rho,\gamma_B]\times E)\\ &+\varepsilon([\gamma_\rho,\gamma_B]\times E)\,\Gamma(\gamma_A\times E)\}\end{aligned}\tag{29}$$

for the quantity

$$\Gamma^{(\varepsilon)}_{AB}\equiv(1/2)\,(\Gamma_{A:B}+\varepsilon\Gamma_{B:A})\tag{30}$$

with

$$\Gamma^{(\varepsilon)+}_{AB}=\Gamma^{(\varepsilon)}_{BA}=\varepsilon\Gamma^{(\varepsilon)}_{AB}.$$

For the case $\gamma_B=E$, eq. (29) reduces dropping the index B to

$$\begin{aligned}[\beta_\rho,\Gamma^{(\varepsilon)}_{k}]&=-i\Gamma^{(\varepsilon)}_{[\rho k]},\\ [\beta_\rho,\Gamma^{(\varepsilon)}_{[mn]}]&=i(\delta_{\rho m}\Gamma^{(\varepsilon)}_{n}-\delta_{\rho n}\Gamma^{(\varepsilon)}_{m})\end{aligned}\tag{31}$$

with the aid of (17).

Clearly, Γ corresponds to $R_5(0, 0)$ in the first bracket of the right-hand side of (9). The 2×5 elements $\Gamma^{(\varepsilon)}_{5}$ and $\Gamma^{(\varepsilon)}_{[\mu5]}$ appearing in (31) correspond to the two $R_5(1, 0)$, and 2×10 elements $\Gamma^{(\varepsilon)}_{\mu}$ and $\Gamma^{(\varepsilon)}_{[\mu\nu]}$ correspond to the two $R_5(1, 1)$ in the second bracket of (9), hence giving the same scalar and vector theories as in the Duffin-Kemmer theory.

Next let us consider the 15^2 elements wherein neither γ_A nor γ_B is E. The following commutation relations are easily obtained for them:

$$
\begin{aligned}
&[\beta_\rho, \Gamma^{(\varepsilon)}_{mn}] = -i(\Gamma^{(\varepsilon)}_{[\rho m]n} + \Gamma^{(\varepsilon)}_{m[\rho n]}) = -i(\Gamma^{(\varepsilon)}_{m[\rho n]} + \varepsilon\Gamma^{(\varepsilon)}_{n[\rho m]}),\\
&[\beta_\rho, \Gamma^{(\varepsilon)}_{j[mn]}] = -i\Gamma^{(\varepsilon)}_{[\rho j][mn]} + i(\delta_{\rho m}\Gamma^{(\varepsilon)}_{jn} - \delta_{\rho n}\Gamma^{(\varepsilon)}_{jm}),\\
&[\beta_\rho, \Gamma^{(\varepsilon)}_{[jk][mn]}] = i\{\delta_{\rho j}\Gamma^{(\varepsilon)}_{k[mn]} - \delta_{\rho k}\Gamma^{(\varepsilon)}_{j[mn]} + \delta_{\rho m}\Gamma^{(\varepsilon)}_{[jk]n} - \delta_{\rho n}\Gamma^{(\varepsilon)}_{[jk]m}\} \qquad (32)\\
&\quad = i\{(\delta_{\rho j}\Gamma^{(\varepsilon)}_{k[mn]} - \delta_{\rho k}\Gamma^{(\varepsilon)}_{j[mn]}) + \varepsilon(\delta_{\rho m}\Gamma^{(\varepsilon)}_{n[jk]} - \delta_{\rho n}\Gamma^{(\varepsilon)}_{m[jk]})\}.
\end{aligned}
$$

In order to have the elements corresponding to $R_5(1, 1) \times R_5(1, 1)$ in (9), we define, restricting the indices to the range from 1 to 4, as

$$
\begin{aligned}
&U^{(\varepsilon)}_{\mu\nu} = i\Gamma^{(\varepsilon)}_{\mu\nu},\\
&V^{(\varepsilon)}_{\mu[\nu\lambda]} = \Gamma^{(\varepsilon)}_{\mu[\nu\lambda]}, \qquad (33)\\
&-i\Gamma^{(\varepsilon)}_{[\mu\sigma][\nu\lambda]} = W^{(\varepsilon)}_{[\mu\sigma][\nu\lambda]}.
\end{aligned}
$$

There are $10+24+21=55$ elements for $\varepsilon=1$ and $6+24+15=45$ for $\varepsilon=-1$, hence 100 elements in all. The commutation relations for them read as follows:

$$
\begin{aligned}
&[\beta_\rho, U^{(\varepsilon)}_{\mu\nu}] = V^{(\varepsilon)}_{\nu[\rho\mu]} + \varepsilon V^{(\varepsilon)}_{\mu[\rho\nu]},\\
&[\beta_\rho, V^{(\varepsilon)}_{\mu[\nu\lambda]}] = W^{(\varepsilon)}_{[\rho\mu][\nu\lambda]} + \delta_{\rho\nu}U^{(\varepsilon)}_{\mu\lambda} - \delta_{\rho\lambda}U^{(\varepsilon)}_{\mu\nu}, \qquad (34)\\
&[\beta_\rho, W^{(\varepsilon)}_{[\mu\sigma][\nu\lambda]}] = \delta_{\rho\mu}V^{(\varepsilon)}_{\sigma[\nu\lambda]} - \delta_{\rho\sigma}V^{(\varepsilon)}_{\mu[\nu\lambda]}\\
&\qquad\qquad + \varepsilon(\delta_{\rho\nu}V^{(\varepsilon)}_{\lambda[\mu\sigma]} - \delta_{\rho\lambda}V^{(\varepsilon)}_{\nu[\mu\sigma]}).
\end{aligned}
$$

These are the equations (38) in I, only $V^{(\varepsilon)}_{\mu[\nu\lambda]}$ having the opposite sign.

Proceeding in the same way we can obtain the elements corresponding to $R_5(1, 0) \times R_5(1, 0)$ in (9) by putting two of the indices of each element in (32) equal to 5:

$$
\begin{aligned}
&[\beta_\rho, i\Gamma^{(\varepsilon)}_{55}] = (1+\varepsilon)\Gamma^{(\varepsilon)}_{5[\rho\tilde{5}]},\\
&[\beta_\rho, \Gamma^{(\varepsilon)}_{5[\mu\tilde{5}]}] = -i\Gamma^{(\varepsilon)}_{[\rho 5][\mu\tilde{5}]} - i\delta_{\rho\mu}\Gamma^{(\varepsilon)}_{5\tilde{5}}, \qquad (35)\\
&[\beta_\rho, -i\Gamma^{(\varepsilon)}_{[\mu 5][\nu\tilde{5}]}] = \delta_{\rho\mu}\Gamma^{(\varepsilon)}_{5[\nu\tilde{5}]} + \varepsilon\delta_{\rho\nu}\Gamma^{(\varepsilon)}_{5[\mu\tilde{5}]}.
\end{aligned}
$$

If we introduce symbols

$$
\begin{aligned}
&K^{(\varepsilon)} = i\Gamma^{(\varepsilon)}_{55},\\
&L^{(\varepsilon)}_\mu = \Gamma^{(\varepsilon)}_{5[\mu\tilde{5}]}, \qquad (36)\\
&M^{(\varepsilon)}_{\mu\nu} = -i\Gamma^{(\varepsilon)}_{[\mu 5][\nu\tilde{5}]},
\end{aligned}
$$

we get the same equations as eq. (35) in I:

$$
\begin{aligned}
&[\beta_\rho, K^{(\varepsilon)}] = (1+\varepsilon)L^{(\varepsilon)}_\rho,\\
&[\beta_\rho, L^{(\varepsilon)}_\mu] = M^{(\varepsilon)}_{\rho\mu} + \delta_{\rho\mu}K^{(\varepsilon)}, \qquad (37)\\
&[\beta_\rho, M^{(\varepsilon)}_{\mu\nu}] = \delta_{\rho\mu}L^{(\varepsilon)}_\nu + \varepsilon\delta_{\rho\nu}L^{(\varepsilon)}_\mu.
\end{aligned}
$$

The number of elements in (36) is $1+4+10=15$ for $\varepsilon=1$ and $0+4+6=10$ for $\varepsilon=-1$, so that there are $5^2=25$ elements in all.

As was shown in (11), $R_5(1, 1) \times R_5(1, 1)$ contains $R_5(1, 0) \times R_5(1, 0)$. The latter can be extracted when one carries out the contraction of two indices in the

equations (33) and (34). That is, putting

$$\begin{aligned} U^{(\varepsilon)} &= \partial^{\mu\nu} U^{(\varepsilon)}_{\mu\nu}, \\ V^{(\varepsilon)}_{\mu} &= \partial^{\rho\lambda} V^{(\varepsilon)}_{\rho[\mu\lambda]}, \\ W^{(\varepsilon)}_{\mu\nu} &= \partial^{\rho\lambda} W^{(\varepsilon)}_{[\mu\rho][\nu\lambda]} - U^{(\varepsilon)}_{\mu\nu}, \end{aligned} \tag{38}$$

one can easily ascertain that these elements have the same properties as those defined by (36), especially the same commutation relations with β_ρ.

Thus we obtained all the elements in the diagonal part of (9), but there still remains the off-diagonal part $R_5(1, 1) \times R_5(1, 0)$. In order to have it, one has only to put one of the indices in (32) equal to 5. If we put

$$\begin{aligned} R^{(\varepsilon)}_{\mu} &= i\Gamma'^{(\varepsilon)}_{5\mu}, \\ S^{(\varepsilon)}_{[\nu\lambda]} &= \Gamma'^{(\varepsilon)}_{5[\nu\lambda]}, \\ S^{(\varepsilon)}_{\mu\nu} &= \Gamma'^{(\varepsilon)}_{[\mu 5]\nu}, \\ T^{(\varepsilon)}_{\mu[\nu\lambda]} &= -i\Gamma'^{(\varepsilon)}_{[\mu 5][\nu\lambda]}, \end{aligned} \tag{39}$$

then the commutation relations are written as

$$\begin{aligned} [\beta_\rho, R^{(\varepsilon)}_{\mu}] &= S^{(\varepsilon)}_{[\rho\mu]} + S^{(\varepsilon)}_{\rho\mu}, \\ [\beta_\rho, S^{(\varepsilon)}_{[\nu\lambda]}] &= T^{(\varepsilon)}_{\rho[\nu\lambda]} + \delta_{\rho\nu} R^{(\varepsilon)}_{\lambda} - \delta_{\rho\lambda} R^{(\varepsilon)}_{\nu}, \\ [\beta_\rho, S^{(\varepsilon)}_{\mu\nu}] &= T^{(\varepsilon)}_{\mu[\rho\nu]} + \delta_{\rho\mu} R^{(\varepsilon)}_{\nu}, \\ [\beta_\rho, T^{(\varepsilon)}_{\mu[\nu\lambda]}] &= \delta_{\rho\mu} S^{(\varepsilon)}_{[\nu\lambda]} + \delta_{\rho\nu} S^{(\varepsilon)}_{\mu\lambda} - \delta_{\rho\lambda} S^{(\varepsilon)}_{\mu\nu}. \end{aligned} \tag{40}$$

The total number of elements in (39) is $4+6+16+24=50=5\times 10$ for both cases $\varepsilon=1$ and $\varepsilon=-1$.

The next thing to do is to carry out the reductions of the elements obtained above in accordance with the decompositions into irreducible representations of the rotation group in five dimensions [see (10) — (12)]. In doing this we can also establish the correspondence of the newly obtained elements to representations in four dimensions as given in (5) and (13). In the first place let us consider the decomposition (10). The 10 elements of (36) with $\varepsilon=-1$ obviously correspond to $R_5(1, 1)$. Since the 15 elements with $\varepsilon=+1$ correspond to $R_5(2, 0)+R_5(0, 0)$, it is necessary to separate $R_5(0, 0)$. According to (37), $\partial^{\mu\nu} M^{(+)}_{\mu\nu} - K^{(+)}$ commutes with β_μ so that this corresponds to $R_5(0, 0)$. Suppose that after separating this element we get K', L'_μ and $M'_{\mu\nu}$ instead of the original $K^{(+)}$, $L^{(+)}_\mu$ and $M^{(+)}_{\mu\nu}$, then we should have

$$\partial^{\mu\nu} M'_{\mu\nu} - K' = 0 \tag{41}$$

and this condition reduces the number of independent elements to 14, yielding $R_5(2, 0)$. In other words $M'_{\mu\nu}$ now gives $10-1=9$ independent elements, thus corresponding to $R_4(2, 0)$. The four L'_μ and one K' naturally correspond to $R_4(1, 0)$ and $R_4(0, 0)$ [see (13·3) and (14·3)]. One can construct the primed elements as follows. The com-

mutation relations (37) do not change when one adds $\partial^{\nu\rho}M^{(-)}_{\rho\nu}-K^{(-)}$ in such a way as

$$\begin{aligned} K' &= K^{(-)}+c(\partial^{\rho\nu}M^{(-)}_{\rho\nu}-K^{(-)}), \\ L'_\mu &= L^{(+)}_\mu, \\ M'_{\mu\nu} &= M^{(-)}_{\mu\nu}-c\partial_{\mu\nu}(\partial^{\rho\nu}M^{(-)}_{\rho\nu}-K^{(-)}). \end{aligned} \tag{42}$$

Here c is a number determined to be $1/5$ by the condition (41).

Applying similar arguments one can separate $R_5(1,0)\times R_5(1,0)$ from $R_5(1,1)\times R_5(1,1)$ as in (11). The separated part $R_5(1,0)\times R_5(1,0)$ has already been given in (38). This part is equivalent to K, L_μ and $M_{\mu\nu}$, so that according to the preceding paragraph $\partial^{\mu\nu}W_{\mu\nu}-U$ gives an isolated element and the elements U', V'_μ and $W'_{\mu\nu}$ constructed from U, V_μ and $W_{\mu\nu}$ in the same way as (42) satisfy the commutation relations (37) and the additional condition $\partial^{\mu\nu}W'_{\mu\nu}=U'$. Here we have omitted the superscript $(\mathcal{E})$ for the sake of simplicity and we shall do the same in the following discussions. In order to get the elements corresponding to $R_5(2,2)-R_5(2,1)-R_5(1,0)$ in (11) we first remark that the elements

$$\begin{aligned} K_{\mu\nu} &= (1/3)\,(\delta_{\mu\nu}U'-W'_{\mu\nu}), \\ L_{\mu[\nu\lambda]} &= (1/3)\,(-\delta_{\mu\nu}V'_\lambda+\delta_{\mu\lambda}V'_\nu), \\ M_{[\mu\sigma][\nu\lambda]} &= (1/3)\,(\delta_{\mu\nu}W'_{\sigma\lambda}-\delta_{\mu\lambda}W'_{\sigma\nu}+\delta_{\sigma\lambda}W'_{\mu\nu}-\delta_{\sigma\nu}W'_{\mu\lambda}) \end{aligned} \tag{43}$$

satisfy not only the commutation relations of the higher class but also the additional conditions

$$\begin{aligned} &\partial^{\mu\nu}K_{\mu\nu}=U', \\ &\partial^{\rho\lambda}L_{\rho[\mu\lambda]}=V'_\mu=V_\mu, \\ &\partial^{\sigma\lambda}M_{[\mu\sigma][\nu\lambda]}-K_{\mu\nu}=W'_{\mu\nu}. \end{aligned} \tag{44}$$

What we want to get are the three kinds of elements which would satisfy the commutation relations of the higher class as well as the following conditions for their traces:

$$\begin{aligned} &\partial^{\mu\nu}U'_{\mu\nu}=0, \\ &\partial^{\rho\lambda}V'_{\rho[\mu\lambda]}=0, \\ &\partial^{\sigma\lambda}W'_{[\mu\sigma][\nu\lambda]}=U'_{\mu\nu}. \end{aligned} \tag{45}$$

With the aid of (44) it can be shown that they are given as

$$\begin{aligned} U'_{\mu\nu} &= U_{\mu\nu}-K_{\mu\nu}+(1/20)\,\delta_{\mu\nu}(\partial^{\rho\lambda}W_{\rho\lambda}-U), \\ V'_{\mu[\nu\lambda]} &= V_{\mu[\nu\lambda]}-L_{\mu[\nu\lambda]}, \\ W'_{\mu\sigma][\nu\lambda]} &= W_{[\mu\sigma][\nu\lambda]}-M_{[\mu\sigma][\nu\lambda]} \\ &\quad -(1/20)\,(\delta_{\mu\nu}\delta_{\sigma\lambda}-\delta_{\mu\lambda}\delta_{\nu\sigma})\,(\partial^{\tau\tau}W_{\sigma\tau}-U). \end{aligned} \tag{46}$$

The number of independent elements among $U'_{\mu\nu}$, $V'_{\mu[\nu\lambda]}$ and $W'_{[\mu\sigma][\nu\lambda]}$ obtained through the above procedures turns out to be $9+20+11=40$ for $\varepsilon=+1$ and $6+20+9=35$ for $\varepsilon=-1$. Remembering that this set of elements as a whole corresponds to $R_5(2, 2)+R_5(2, 1)+R_5(1, 0)$ in (11) and comparing the above numbers with (14), we may conclude that the latter 35 elements correspond to $R_5(2, 1)$ and the former 40 elements to $R_5(2, 2)+R_5(1, 0)$. Therefore we must further separate $R_5(1, 0)$ in the case of $\varepsilon=+1$. Considering this case only, we define

$$\begin{aligned} V_{[\mu\nu\lambda]}&=(1/3)\,(V'_{\mu[\nu\lambda]}+V'_{\nu[\lambda\mu]}+V'_{\lambda[\mu\nu]}),\\ W_{[\mu\sigma\nu\lambda]}&=(1/3)\,(W'_{[\mu\sigma][\nu\lambda]}+W'_{[\nu\sigma][\lambda\mu]}+W'_{[\lambda\sigma][\mu\nu]}), \end{aligned} \tag{47}$$

both of them being totally antisymmetric with respect to all the indices in square brackets. They form a closed subset with regard to the commutation relation with β_ρ :

$$\begin{aligned} [\beta_\rho, V_{[\mu\nu\lambda]}]&=W_{[\rho\mu\nu\lambda]},\\ [\beta_\rho, W_{[\mu\sigma\nu\lambda]}]&=\delta_{\rho\mu}V_{[\sigma\nu\lambda]}-\delta_{\rho\sigma}V_{[\mu\nu\lambda]}\\ &\quad+\delta_{\rho\nu}V_{[\mu\sigma\lambda]}-\delta_{\rho\lambda}V_{[\mu\sigma\nu]}. \end{aligned} \tag{48}$$

A glance at the relations (34) is sufficient to see that the subtraction of $V_{[\mu\nu\lambda]}$ and $W_{[\mu\sigma\nu\lambda]}$ from $V'_{\mu[\nu\lambda]}$ and $W'_{[\mu\sigma][\nu\lambda]}$ will not affect the commutation relations, $U'_{\mu\nu}$ of course being unchanged. Hence, taking $U'_{\mu\nu}$, $V'_{\mu[\nu\lambda]}$ and $W'_{[\mu\sigma][\nu\lambda]}$ to be the ones from which the totally antisymmetric elements have already been subtracted, we have the additional conditions for them :

$$\begin{aligned} V'_{\mu[\nu\lambda]}+V'_{\nu[\lambda\mu]}+V'_{\lambda[\mu\nu]}&=0,\\ W'_{[\mu\sigma][\nu\lambda]}+W'_{[\nu\sigma][\lambda\mu]}+W'_{[\lambda\sigma][\mu\nu]}&=0. \end{aligned} \tag{49}$$

Consequently the number of independent elements among them becomes $9+16+10=35$ in agreement with $(14\cdot1)$. Thus we have finally obtained $R_5(2, 2)$, separating $R_5(1, 0)$ as $V_{[\mu\nu\lambda]}$ and $W_{[\mu\sigma\nu\lambda]}$.

Also for the case of $\varepsilon=-1$ we can subtract from 20 elements $V'_{\mu[\nu\lambda]}$ the totally antisymmetric elements which transform as pseudo-vector for four-dimensional rotations, consequently giving $R_4(1, 0)$. The 16 elements left after the subtraction give $R_4(2, 1)$. The discussion for this point will be given below (92).

There remains the reduction of the off-diagonal part (39). As was shown in (12) we must extract $R_5(2, 1)$ separating $R_5(1, 1)$ and $R_5(1, 0)$ off. Since the equations (40) do not contain ε explicitly, the following discussion applies equally to both cases. $R_5(1, 0)$ can be separated as

$$S=\partial^{\mu\nu}S_{\mu\nu}, \quad S_\mu=R_\mu+\partial^{\rho\lambda}T_{\rho[\mu\lambda]}, \tag{50}$$

for which we have

$$[\beta_\rho, S]=S_\rho, \quad [\beta_\rho, S_\mu]=\delta_{\rho\mu}S. \tag{51}$$

If one defines totally antisymmetric elements by

$$T_{[\mu\nu]}=(1/3)\,(S_{[\mu\nu]}-S_{\mu\nu}+S_{\nu\mu}),$$
$$T_{[\mu\nu\lambda]}=(1/3)\,(T_{\mu[\nu\lambda]}+T_{\nu[\lambda\mu]}+T_{\lambda[\mu\nu]}), \tag{52}$$

they satisfy

$$[\beta_\rho,\ T_{[\mu\nu]}]=T_{[\rho\mu\nu]},$$
$$[\beta_\rho,\ T_{[\mu\nu\lambda]}]=\delta_{\rho\mu}T_{[\nu\lambda]}+\delta_{\rho\nu}T_{[\lambda\mu]}+\delta_{\rho\lambda}T_{[\mu\nu]}. \tag{53}$$

This set of 10 elements is $R_5(1,\ 1)$. In order to separate these from the rest of elements we introduce

$$T'_{\mu[\nu\lambda]}=T_{\mu[\nu\lambda]}-T_{[\mu\nu\lambda]}+(1/4)\,(\delta_{\mu\nu}S_\lambda-\delta_{\mu\lambda}S_\nu),$$
$$S'_{[\mu\nu]}=S_{[\mu\nu]}-T_{[\mu\nu]},$$
$$S'_{\mu\nu}=S_{\mu\nu}+T_{[\mu\nu]}-(1/4)\,\delta_{\mu\nu}S, \tag{54}$$
$$R'_\mu=R_\mu-(1/4)\,S_\mu.$$

It can be shown that they satisfy the same commutation relation as (40) and at the same time obey the additional conditions

$$\partial^{\mu\nu}S'_{\mu\nu}=0,\quad R'_\mu-\partial^{\nu\rho}T'_{\nu\rho\mu}=0,$$
$$S'_{[\mu\nu]}=S'_{\mu\nu}-S'_{\nu\mu}, \tag{55}$$
$$T'_{[\mu\nu\lambda]}+T'_{[\nu\lambda\mu]}+T'_{[\lambda\mu\nu]}=0,$$

so that $T'_{\mu[\nu\lambda]}$ and R'_μ have 16 and 4 independent elements respectively and $S'_{\mu\nu}$ gives 15 consisting of 6 antisymmetric $S'_{[\mu\nu]}$ and 9 symmetric ones. This is in agreement with (14·2). Hence they provide $R_5(2,\ 1)$.

§ 3. Derivation and analytical reduction of tensor equations

Since we have obtained the complete sets of linearly independent elements each of which corresponds to an irreducible representation of rotation group in five dimensions and is at the same time closed with respect to the commutation relation with ${}_i\beta_\mu$, we can readily write down the wave equation (1) in the form of field equations for tensor quantities. Then we shall reduce each set of them to more fundamental equations for simple fields each corresponding to definite values of spin and mass.

In (8) we may assume without loss of generality that ${}_i\beta'_\mu$ is the negative transpose of ${}_i\beta_\mu$. Then we can rewrite (1) as an equation for the 16×16 matrix function Ψ whose components are those of the original ψ rearranged in a suitable way:

$$\partial^\mu[{}_i\beta_\mu,\ \Psi]-\kappa\Psi=0. \tag{56}$$

We define field quantities by

$$\psi^{(s)}_{AB}=\mathrm{sp}\,(\Gamma^{(s)}_{AB}\Psi). \tag{57}$$

Evidently $\psi^{(s)}_{AB}$ possesses the same transformation character as $\Gamma^{(s)}_{AB}$. The field equations

for these functions are

$$\partial^\rho \operatorname{sp}([\beta_\rho, \Gamma^{(\varepsilon)}_{AB}]\Psi) + \kappa \operatorname{sp}(\Gamma^{(\varepsilon)}_{AB}\Psi) = 0, \tag{58}$$

which results from the simple procedures of multiplying (56) by $\Gamma'^{(\varepsilon)}_{AB}$ and taking spur.

If we take as $\Gamma'^{(\varepsilon)}_{AB}$ the elements $\Gamma : \Gamma'^{(\varepsilon)}_{5}$, $\Gamma'^{(\varepsilon)}_{[\mu 5]} : \Gamma'^{(\varepsilon)}_{\mu}$, $\Gamma'^{(\varepsilon)}_{\mu\nu}$, the equation (58) will obviously yield trivial scalar, scalar and vector theories respectively. Next consider the elements in the diagonal part corresponding to $R_5(1, 0) \times R_5(1, 0)$. $\partial^{\mu\nu} M_{\mu\nu} - K$ gives a trivial scalar and K', L'_μ and $M'_{\mu\nu}$ give, when substituted in (58), the following set of equations with the aid of (37) :

$$\begin{aligned} &(1+\varepsilon)\,\partial^\rho \psi'_\rho + \kappa\varphi = 0, \\ &\partial^\rho \psi_{\rho\lambda} + \partial_\lambda \varphi + \kappa \psi'_\lambda = 0, \\ &\partial_\lambda \psi'_\rho + \varepsilon \partial_\rho \psi'_\lambda + \kappa \psi'_{\lambda\rho} = 0. \end{aligned} \tag{59}$$

Owing to (41) there is an additional condition

$$\delta^{\lambda\rho}\psi_{\lambda\rho} = \varphi. \tag{60}$$

For $\varepsilon = -1$ this set of field equations is simply the equations for vector meson with mass κ. For $\varepsilon = +1$ it can be split up into those for a scalar field with mass $\kappa/2$ and for a vector field with mass κ in the following way. First, (59) give

$$[\Box - (\kappa/2)^2]\varphi = 0, \tag{61}$$

so that an auxiliary vector field φ_ρ can be introduced in such a way that

$$2\partial^\rho \varphi_\rho + \kappa\varphi = 0, \tag{62}$$

$$2\partial_\rho \varphi + \kappa \varphi_\rho = 0. \tag{62'}$$

Then the tensor quantity $\varphi_{\lambda\rho}$ defined by

$$\partial_\lambda \varphi_\rho + \partial_\rho \varphi_\lambda + \kappa \varphi_{\lambda\rho} = 0 \tag{63}$$

satisfy the equations

$$\partial^\rho \varphi_{\rho\lambda} + \partial_\lambda \varphi + \kappa \varphi_\lambda = 0 \tag{64}$$

and

$$\delta^{\lambda\rho}\varphi_{\lambda\rho} = \varphi. \tag{65}$$

The four equations (62), (64), (63) and (65) are nothing but the above (59) and (60) wherein ψ'_ρ and $\psi'_{\lambda\rho}$ are replaced by φ_ρ and $\varphi_{\lambda\rho}$ respectively. Hence we immediately get the equations

$$\begin{aligned} &2\partial^\rho \chi_\rho = 0, \\ &\partial^\rho \chi_{\rho\lambda} + \kappa\chi_\lambda = 0, \\ &\partial_\lambda \chi_\rho + \partial_\rho \chi_\lambda + \kappa \chi_{\lambda\rho} = 0 \end{aligned} \tag{59'}$$

and

$$\delta^{\lambda\rho}\chi_{\lambda\rho} = 0 \tag{60'}$$

for the quantities defined by

$$\chi_\rho=\psi_\rho-\varphi_\rho,\quad \chi_{\lambda\rho}=\psi_{\lambda\rho}-\varphi_{\lambda\rho}, \tag{66}$$

which evidently yield the field of spin 1 with mass κ.

In the same way one can easily obtain the field equations of higher class in the diagonal part corresponding to $R_5(1,1)\times R_5(1,1)$. From U', V'_μ and $W'_{\mu\nu}$ referred to just before (43) we get the same equations as those of the lower class (59). The totally antisymmetric elements (47) which could be separated in the case of $\varepsilon=+1$ give the following set of equations:

$$\begin{aligned}&\partial^\rho\chi_{[\rho\mu\nu\lambda]}+\kappa\chi_{[\mu\nu\lambda]}=0,\\ &\partial_\mu\chi_{[\rho\nu\lambda]}-\partial_\rho\chi_{[\mu\nu\lambda]}+\partial_\nu\chi_{[\mu\rho\lambda]}-\partial_\lambda\chi_{[\mu\nu\rho]}+\kappa\chi_{[\mu\nu\rho\lambda]}=0.\end{aligned} \tag{67}$$

Eliminating $\chi_{[\mu\nu\lambda]}$ we get

$$(\Box-\kappa^2)\chi_{[\mu\nu\lambda]}=0,\quad \partial^\rho\chi_{[\mu\nu\rho]}=0, \tag{68}$$

so that we have a scalar field with mass κ. The rest of the elements corresponding to $R_5(2,1)$ for $\varepsilon=-1$ and $R_5(2,2)$ for $\varepsilon=+1$ furnish the equations of the higher class:

$$\begin{aligned}&\partial^\rho(\psi_{\mu[\nu\rho]}+\varepsilon\psi_{\nu[\mu\rho]})+\kappa\varphi_{\mu\nu}=0,\\ &\partial^\rho\psi_{[\mu\rho][\nu\lambda]}+\partial_\lambda\varphi_{\mu\nu}-\partial_\nu\varphi_{\mu\lambda}+\kappa\psi_{[\mu[\nu\lambda]]}=0,\\ &\partial^\rho\psi_{\mu[\nu\lambda]}-\partial_\mu\psi_{\rho[\nu\lambda]}+\varepsilon(\partial_\lambda\psi_{\nu[\mu\rho]}-\partial_\nu\psi_{\lambda[\mu\rho]})+\kappa\psi_{[\mu\rho][\nu\lambda]}=0.\end{aligned} \tag{69}$$

On these field quantities are imposed the additional conditions resulting from (45)

$$\begin{aligned}&\partial^{\mu\nu}\varphi_{\mu\nu}=0,\\ &\partial^{\mu\nu}\psi_{\mu[\nu\rho]}=0,\\ &\partial^{\mu\nu}\psi_{[\mu\rho][\nu\lambda]}=\varphi_{\rho\lambda}\end{aligned} \tag{70}$$

and moreover in the case $\varepsilon=+1$ the conditions

$$\begin{aligned}&\psi_{\mu[\nu\lambda]}+\psi_{\nu[\lambda\mu]}+\psi_{\lambda[\mu\nu]}=0,\\ &\psi_{[\mu\rho][\nu\lambda]}+\psi_{[\nu\rho][\lambda\mu]}+\psi_{[\lambda\rho][\mu\nu]}=0.\end{aligned} \tag{71}$$

The reduction of the field equations (69) proceeds in the following way. In the first place the first and the second of (69) turn out to be

$$\partial^\mu\hat{\xi}_{[\nu\mu]}+\kappa\hat{\xi}_\nu=0\quad\text{and}\quad\partial_\nu\hat{\xi}_\mu-\partial_\mu\hat{\xi}_\nu+\kappa\hat{\xi}_{[\mu\nu]}=0 \tag{72}$$

for the quantities $\hat{\xi}_\mu$ and $\hat{\xi}_{[\mu\nu]}$ defined by

$$\hat{\xi}_\mu=(1/\kappa)\,\partial^\rho\varphi_{\rho\mu}\quad\text{and}\quad\hat{\xi}_{[\mu\nu]}=(1/\kappa)\,\partial^\rho\psi_{\rho[\mu\nu]}, \tag{73}$$

which yield a vector field of mass κ. Then for the quantity defined by

$$\eta_{\mu\nu}=\varphi_{\mu\nu}-(1/\kappa)\,(\partial_\mu\hat{\xi}_\nu+\varepsilon\partial_\nu\hat{\xi}_\mu) \tag{74}$$

we have [see the equations below (69) in I]

$$[\Box-(\kappa/2)^2]\eta_{\mu\nu}=0, \tag{75}$$

and

$$\partial^{\mu\nu}\eta_{\mu\nu}=0, \quad \partial^{\mu}\eta_{\mu\nu}=0. \tag{76}$$

The divergence condition suppresses one of the four indices $\rho=1$, 2, 3 and 4, so that $\eta_{\mu\nu}$ may be regarded as a tensor in 3-dimensional space. The number of independent elements of antisymmetric $\eta_{[\mu\nu]}$ is 3 for $\varepsilon=-1$, thus giving a vector theory, and that of symmetric $\eta_{(\mu\nu)}$ is 6 for $\varepsilon=+1$, which is further reduced by one by $\partial^{\mu\nu}\eta_{(\mu\nu)}=0$ giving a theory of spin 2. The mass value is evidently $\kappa/2$ for both cases. In the second place by introducing the quantity

$$\begin{aligned}\varphi_{\mu[\nu\lambda]}=&(3/2\kappa)\,\partial_\mu\xi_{[\nu\lambda]}-(1/2)\,(\delta_{\mu\nu}\xi_\lambda-\delta_{\mu\lambda}\xi_\nu)\\&+(2/\kappa)\,(\partial_\nu\eta_{\mu\lambda}-\partial_\lambda\eta_{\mu\nu}),\end{aligned} \tag{77}$$

we can reproduce all the equations in (69) and (70) wherein $\psi_{\mu[\nu\lambda]}$ and $\psi_{[\mu\rho][\nu\lambda]}$ are replaced respectively by $\varphi_{\mu[\nu\lambda]}$ and $\varphi_{[\mu\rho][\nu\lambda]}$ defined by the equation of the same structure as the third of (69). Hence the quantities

$$\begin{aligned}\zeta_{\mu[\nu\lambda]}&=\psi'_{\mu[\nu\lambda]}-\varphi_{\mu[\nu\lambda]},\\ \zeta_{[\mu\rho][\nu\lambda]}&=\psi'_{[\mu\rho][\nu\lambda]}-\varphi_{[\mu\rho][\nu\lambda]}\end{aligned} \tag{78}$$

satisfy the field equations

$$\begin{aligned}&\partial^\mu(\zeta_{\mu[\nu\rho]}+\varepsilon\zeta_{\nu[\mu\rho]})=0,\\ &\partial^\rho\zeta_{[\mu\rho][\nu\lambda]}+\kappa\zeta_{\mu[\nu\lambda]}=0,\\ &\partial_\rho\zeta_{\mu[\nu\lambda]}-\partial_\mu\zeta_{\rho[\nu\lambda]}+\varepsilon\,(\partial_\lambda\zeta_{\nu[\mu\rho]}-\partial_\nu\zeta_{\lambda[\mu\rho]})+\kappa\zeta_{[\mu\rho][\nu\lambda]}=0,\end{aligned} \tag{79}$$

and the additional conditions

$$\partial^{\mu\nu}\zeta_{\mu[\nu\lambda]}=0, \quad \partial^{\mu\nu}\zeta_{[\mu\rho][\nu\lambda]}=0. \tag{80}$$

Then the equations (79) yield [see below (73) in I]

$$\partial^\mu\zeta_{\mu[\nu\rho]}=\partial^\rho\zeta_{\mu[\nu\rho]}=0. \tag{81}$$

Furthermore the third of (79) is rewritten as

$$\begin{aligned}&\partial_\rho\zeta_{\mu[\nu\lambda]}-\partial_\mu\zeta_{\rho[\nu\lambda]}+\partial_\lambda\zeta_{\nu[\mu\rho]}-\partial_\nu\zeta_{\lambda[\mu\rho]}\\&\quad+\kappa\{\zeta_{[\mu\rho][\nu\lambda]}+(1/\kappa)\,(1-\varepsilon)\,(\partial_\nu\zeta_{\lambda[\mu\rho]}-\partial_\lambda\zeta_{\nu[\mu\rho]})\}=0.\end{aligned} \tag{82}$$

Summing up three equations given by cyclic permutation of μ, ν and λ in (82) we get the second of (67) with

$$\chi_{[\mu\nu\lambda]}=(1/3)\,(\zeta_{\mu[\nu\lambda]}+\zeta_{\nu[\lambda\mu]}+\zeta_{\lambda[\mu\nu]}) \tag{83}$$

and

$$\chi_{[\mu\rho\nu\lambda]}=(1/3)\left\{\begin{pmatrix}\zeta_{[\mu\rho][\nu\lambda]}\\+\zeta_{[\nu\rho][\lambda\mu]}\\+\zeta_{[\lambda\rho][\mu\nu]}\end{pmatrix}+(1/\kappa)\,(1-\varepsilon)\begin{pmatrix}\partial_\nu\zeta_{\lambda[\mu\rho]}-\partial_\lambda\zeta_{\nu[\mu\rho]}\\+\partial_\lambda\zeta_{\mu[\nu\rho]}-\partial_\mu\zeta_{\lambda[\nu\rho]}\\+\partial_\mu\zeta_{\nu[\lambda\rho]}-\partial_\nu\zeta_{\mu[\lambda\rho]}\end{pmatrix}\right\}. \tag{84}$$

By definition $\chi_{[\mu\nu\lambda]}$ is totally antisymmetric in indices and the second of (67) assures that this holds also for $\chi_{[\mu\rho\nu\lambda]}$. Then (81) and the second of (79) give the first of (67). We thus get a scalar field with mass κ. Now for the fields

$$\theta_{\mu[\nu\lambda]}=\zeta_{\mu[\nu\lambda]}-\chi_{\mu[\nu\lambda]}, \tag{85}$$

$$\theta_{[\mu\rho][\nu\lambda]}=\zeta_{[\mu\rho][\nu\lambda]}+(1/\kappa)\,(1-\varepsilon)\,(\partial_\nu\zeta_{\lambda[\mu\rho]}-\partial_\lambda\zeta_{\nu[\mu\rho]})-\chi_{[\mu\rho\nu\lambda]} \tag{86}$$

we have the equations

$$\begin{aligned}&\partial^\rho\theta_{[\mu\rho][\nu\lambda]}+\kappa\theta_{\mu[\nu\lambda]}=0,\\ &\partial_\rho\theta_{\mu[\nu\lambda]}-\partial_\mu\theta_{\rho[\nu\lambda]}+\partial_\lambda\theta_{\nu[\mu\rho]}-\partial_\nu\theta_{\lambda[\mu\rho]}+\kappa\theta_{[\mu\rho][\nu\lambda]}=0\end{aligned} \tag{87}$$

with the additional conditions

$$\begin{aligned}&\theta_{\mu[\nu\lambda]}+\theta_{\nu[\lambda\mu]}+\theta_{\lambda[\mu\nu]}=0,\\ &\theta_{[\mu\rho][\nu\lambda]}+\theta_{[\nu\rho][\lambda\mu]}+\theta_{[\lambda\rho][\mu\nu]}=0.\end{aligned} \tag{88}$$

$\theta_{\mu[\nu\lambda]}$ is a field with mass κ and the additional conditions

$$\partial^\mu\theta_{\mu[\nu\lambda]}=\partial^\lambda\theta_{\mu[\nu\lambda]}=0 \tag{89}$$

and

$$\partial^{\mu\nu}\theta_{\mu[\nu\lambda]}=0. \tag{90}$$

The condition (89) tells us that the number of independent components of $\theta_{\mu[\nu\lambda]}$ is 9, but the conditions (90) and (88) reduce it by 3 and by one respectively meaning that the field $\theta_{\mu[\nu\lambda]}$ is of spin 2.

Since we have

$$\varphi_{\mu[\nu\lambda]}+\varphi_{\nu[\lambda\mu]}+\varphi_{\lambda[\mu\nu]}=0, \tag{91}$$

the $\chi_{[\mu\nu\lambda]}$ defined by (83) is nothing but $(1/3)(\zeta^\rho{}_{\rho\mu\nu}-\zeta^\rho{}_{\nu\mu\rho}-\zeta^\rho{}_{\mu\nu\rho})$ irrespective of the value of ε. Then for $\varepsilon=+1$ alone we have the condition

$$\varphi_{[\mu\rho][\nu\lambda]}+\varphi_{[\nu\rho][\lambda\mu]}+\varphi_{[\lambda\rho][\mu\nu]}=0 \tag{92}$$

and the $\chi_{[\mu\rho\nu\lambda]}$ given by (84) is rewritten as $(1/3)(\zeta^\rho{}_{\rho\mu\nu}-\zeta^\rho{}_{\nu\mu\rho}+\zeta^\rho{}_{\mu\nu\rho})$. The definitions of $\chi_{[\mu\nu\lambda]}$ and $\chi_{[\mu\rho\nu\lambda]}$ thus contain no derivatives, so that these can be algebraically separated off from the original fields. The separated quantities are already given in (67) and accordingly we have the condition (71), which entails the identical vanishing of $\chi_{[\mu\nu\lambda]}$ and $\chi_{[\mu\rho\nu\lambda]}$ in (83) and (84). In other words the $\theta_{\mu[\nu\lambda]}$ is identical with the $\zeta_{\mu[\nu\lambda]}$. Contrary to this the definition (84) contains derivative terms for $\varepsilon=-1$. Therefore the reduction of the field $\zeta_{\mu[\nu\lambda]}$ into $\chi_{[\mu\nu\lambda]}$ and $\theta_{\mu[\nu\lambda]}$ cannot be algebraical, but is an analytical one [see below (49)].

In the case of reduction of the off-diagonal part there is no need of separate treatment of the two cases $\varepsilon=\pm1$. From (50) and (51) we get a scalar theory. The totally antisymmetric elements (52) yield with the aid of (53)

$$\begin{aligned}&\partial^\rho\varphi_{[\rho\mu\nu]}+\kappa\varphi_{[\mu\nu]}=0,\\ &\partial_\mu\varphi_{[\nu\lambda]}+\partial_\nu\varphi_{[\lambda\mu]}+\partial_\lambda\varphi_{[\mu\nu]}+\kappa\varphi_{[\mu\nu\lambda]}=0,\end{aligned} \tag{93}$$

whence the equations for a vector field with mass κ can be derived:

$$(\Box-\kappa^2)\,\varphi_{[\mu\nu]}=0, \quad \partial^\mu\varphi_{[\mu\nu]}=0. \tag{94}$$

The field equations derived from (54) with the aid of the commutation relations (40) need some analysis. They are

$$\partial^\rho\psi_{[\rho\lambda]}+\partial^\rho\psi_{\rho\lambda}+\kappa\varphi_\lambda=0, \tag{95}$$

$$\partial_\mu\varphi_\nu-\partial^\rho\psi_{\mu[\nu\rho]}+\kappa\psi_{\mu\nu}=0, \tag{96}$$

$$\partial^\rho\psi_{\rho[\mu\nu]}+\partial_\mu\varphi_\nu-\partial_\nu\varphi_\mu+\kappa\psi_{[\mu\nu]}=0, \tag{97}$$

$$\partial_\mu\psi_{[\nu\lambda]}+\partial_\nu\psi_{\mu\lambda}-\partial_\lambda\psi_{\mu\nu}+\kappa\psi_{\mu[\nu\lambda]}=0. \tag{98}$$

The additional conditions resulting from (55) are imposed on the field quantities:

$$\delta^{\mu\nu}\psi_{\mu\nu}=0, \tag{99}$$

$$\psi_{[\mu\nu]}=\psi_{\mu\nu}-\psi_{\nu\mu}, \tag{100}$$

$$\delta^{\mu\nu}\psi_{\mu[\nu\rho]}=\varphi_\rho\,. \tag{101}$$

First with the aid of (95) we define a field $\tilde{\xi}_\mu$ by

$$(1/2)\,\tilde{\xi}_\mu=(1/\kappa)\,\partial^\rho\psi_{[\mu\rho]}=(1/\kappa)\,\partial^\rho\psi_{\rho\mu}+\varphi_\mu \tag{102}$$

and in addition we introduce the symbols

$$\chi_\mu=-(1/\kappa)\,\partial^\rho\psi_{\mu\rho}\,, \quad \chi=(1/\kappa)\,\partial^\rho\varphi_\rho\,. \tag{103}$$

Then according to (100) the field φ_μ splits up into two parts:

$$\varphi_\mu=\tilde{\xi}_\mu+\chi_\mu\,. \tag{104}$$

From (102) we have immediately

$$\partial^\rho\tilde{\xi}_\rho=0 \tag{105}$$

and

$$\partial^\rho\chi_\rho-\kappa\chi=0. \tag{106}$$

The field χ furnishes a scalar field with mass κ, because (96) yields

$$\partial_\mu\chi-\kappa\chi_\mu=0. \tag{107}$$

The field $\tilde{\xi}_\mu$ represents a vector field with mass $\kappa/2$, since we have, on eliminating $\partial^\lambda\partial^\rho\psi_{\lambda[\mu\rho]}$ from (97) and using (95),

$$[\Box-(\kappa/2)^2]\tilde{\xi}_\mu=0. \tag{108}$$

Now we introduce

$$\varphi_{\mu\nu}=-(2/\kappa)\,\partial_\mu\tilde{\xi}_\nu-(1/3)\,[(4/\kappa^2)\,\partial_\mu\partial_\nu\chi-\delta_{\mu\nu}\chi] \tag{109}$$

with the additional condition corresponding to (99)

$$\delta^{\mu\nu}\varphi_{\mu\nu}=0. \tag{99'}$$

Moreover we introduce $\varphi_{[\mu\nu]}$ and $\varphi_{\mu[\nu\lambda]}$ respectively in conformity with (100) and (98),

$$\varphi_{[\mu\nu]}=\varphi_{\mu\nu}-\varphi_{\nu\mu}=-(2/\kappa)(\partial_\mu\tilde{\xi}_\nu-\partial_\nu\tilde{\xi}_\mu) \tag{100'}$$

and

$$\begin{aligned}\varphi_{\mu[\nu\lambda]}&=-(1/\kappa)(\partial_\mu\varphi_{[\nu\lambda]}+\partial_\nu\varphi_{\mu\lambda}-\partial_\lambda\varphi_{\mu\nu})\\&=(4/\kappa^2)\partial_\mu(\partial_\nu\tilde{\xi}_\lambda-\partial_\lambda\tilde{\xi}_\nu)+(1/3\kappa)(\delta_{\mu\nu}\partial_\lambda-\delta_{\mu\lambda}\partial_\nu)\chi,\end{aligned}$$

which satisfies

$$\partial^{\mu\nu}\varphi_{\mu[\nu\lambda]}=\varphi_\lambda\,. \tag{101'}$$

Then it is easy to show that these quantities satisfy the equations

$$\partial^\rho\varphi_{\rho\mu\nu}+\partial^\rho\varphi_{\nu\lambda}-\kappa\varphi_{\mu\nu}=0, \tag{95'}$$

$$\partial_\mu\varphi_\nu-\partial^\rho\varphi_{\mu[\nu\rho]}+\kappa\varphi_{\mu\nu}=0, \tag{96'}$$

$$\partial^\rho\varphi_{\rho[\mu\nu]}+\partial_\mu\varphi_\nu-\partial_\nu\varphi_\mu+\kappa\varphi_{[\mu\nu]}=0 \tag{97'}$$

and furthermore to show that the quantities

$$\chi_{\mu\nu}=\psi_{\mu\nu}-\varphi_{\mu\nu}\,, \tag{110}$$

$$\chi_{[\mu\nu]}=\chi_{\mu\nu}-\chi_{\nu\mu}\,, \tag{111}$$

and

$$\chi_{\mu[\nu\lambda]}=\psi_{\mu[\nu\lambda]}-\varphi_{\mu[\nu\lambda]} \tag{112}$$

with the conditions

$$\partial^{\mu\nu}\chi_{\mu\nu}=0,\quad \partial^{\mu\nu}\chi_{\mu[\nu\lambda]}=0 \tag{113}$$

satisfy the equations

$$\partial^\rho\chi_{[\rho\lambda]}+\partial^\rho\chi_{\rho\lambda}=0, \tag{114}$$

$$-\partial^\rho\chi_{\mu[\nu\rho]}+\kappa\chi_{\mu\nu}=0, \tag{115}$$

$$\partial^\rho\chi_{\rho[\mu\nu]}+\kappa\chi_{[\mu\nu]}=0, \tag{116}$$

$$\partial_\mu\chi_{[\nu\lambda]}+\partial_\nu\chi_{\mu\lambda}-\partial_\lambda\chi_{\mu\nu}+\kappa\chi_{\mu[\nu\lambda]}=0. \tag{117}$$

From (115) we get

$$\partial^\nu\chi_{\mu\nu}=0, \tag{118}$$

which yields with the aid of (111) and (114)

$$\partial^\mu\chi_{\mu\nu}=0. \tag{119}$$

Then by (115) and (117) one obtains

$$(\Box-\kappa^2)\chi_{\mu\nu}=0. \tag{120}$$

The mass value of the field is κ. The symmetric part $\chi_{(\mu\nu)}$ and the antisymmetric part $\chi_{[\mu\nu]}$ afford the fields of spin 2 and of spin 1 respectively.

§ 4. Concluding Remarks

With the algebraic and analytical reductions in the preceding sections we have finished the analysis of the relativistic wave equation (1) with maximum spin 2. The simple fields corresponding to definite values of spin and mass, which is peculiar to the theory of maximum spin 2, are listed in the following table. The number attached to each field quantity specifies the formula to be referred to.

Representation	mass/κ	spin	field			
$R_5(2, 0)$	1/2	0	φ	(62)		
	1	1	χ_μ	(66)		
$R_5(2, 2)$	1/2	2	$\eta_{(\mu\nu)}$	(74)		
	1	2	$\theta_{\mu[\nu\lambda]}$	(85)		
		1	ξ_μ	(73)		
$R_5(2, 1)$	1/2	1	$\eta_{[\mu\nu]}$	(74)	ξ_μ	(102)
	1	2	$\theta_{\mu[\nu\lambda]}$	(85)	$\chi_{(\mu\nu)}$	(110)
		1	ξ_μ	(73)	$\chi_{[\mu\nu]}$	(111)
		0	$\chi_{[\mu\nu\lambda]}$	(83)	χ	(103)
			diagonal		off-diagonal	

The expressions for total energy and total charge can easily be found in terms of the reduced field quantities : we may refer to the discussions in I which remain valid with suitable modifications and supplements. It must be noted here that the field with mass value $\kappa/2$ has energy density of the sign opposite to that for mass value κ.

Historically the analysis of the relativistic wave equation (1) under the assumption (8) was first undertaken by Tonnelat[3] as early as in 1941 in order to investigate the interrelationship between the field with spin 2 and the gravitational one. But the mathematical procedures adopted in her works was far from completeness and the same was also true for the recent attack made independently by Green.[4] Now in the present paper we have arrived at the final solution of the problem by eliminating the shortcomings in our preceding paper I. But throughout in these investigations we have been concerned only with the purely mathematical aspects of the problem. We are interested neither in justifying the basic assumptions (1) and (8) on the basis of some realistic models[5] or of purely postulational reasonings[6] nor in applying the mathematical apparatus developed here to the computation of actual processes coupled with the technique of the second quantization. We admit that the object of the present investigation is of extremely a hypothetical character and moreover that the theory would never find a counterpart in the actual physical world, since recent experimental evidences seem to show that the spin values of elementary particles are 0 or 1/2 leaving no room for higher ones. Frankly speaking we had better to do without a very complicated construction such as

developed in this paper. Nevertheless we shall never be able to do without relativistic wave equations for describing the behavior of elementary particles, so the investigation of relativistic wave equations should be made extensively from a point of view as general as possible. In this article our stress is thus laid on the purely mathematical analysis of the problem and at the same time on revealing exhaustively the features implied by the fundamental postulates (1) and (8).

In conclusion the possibility of generalization and extension of the procedures presented in this article may be indicated. This implies the analysis of the fundamental wave equation (1) with the matrix coefficient α_ρ being a direct sum of an arbitrary number of independent Dirac operators, hence yielding the theory of an arbitrary value of maximum spin. From algebraical point of view the wave function should be an irreducible representation of the five-dimensional rotation group and its constituent parts, each being irreducible for the rotation group in four dimensions, will be related with each other by the operation of α_ρ in (1) furnishing a set of relativistically covariant tensor equations. But these sets of algebraically irreducible equations cannot in general be irreducible from the analytical side, in other words, they are to be separated by analytical procedures into simple field equations each corresponding to definite values of spin and mass. For this purpose it is necessary first to discriminate before actual separation what kinds of spin and mass are possible for a particular irreducible representation of five-dimensional rotation group and then to construct projection operators performing automatically the analytical reduction into simple constituents. Furthermore from the physical point of view we should have a straightforward procedure of determining the sign of the contribution from such a simple field to the charge or energy densities. All these involve the complete analysis of the so-called fusion theory of the relativistic wave equations in the sense of de Broglie[7], which we hope will be presented in near future.

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Possible Experimental Tests on the Decay Interactions of Hyperons

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We examine what information could be obtained by the measurement of the characteristic quantities of the $(N+\pi)$-decays of Σ- and Λ-hyperons. In particular, the possibility of experimental tests for the $|\Delta I|=1/2$ rule and the postulate of time-reversal invariance is discussed. We also discuss that the relative magnitudes of the effective strengths of the $|\Delta I|=1/2$ and other (if they exist) interactions could be known under suitable assumptions on the dynamical nature of the interactions. Some physically interesting examples of such assumptions are presented and it is shown that the validity of those assumptions can be tested experimentally. For obtaining such considerably definite information, it is very much desired to measure, at least, the ratio of the asymmetry parameters for various modes of decays.

§ 1. Introduction

Stimulated by the success of the Nishijima and Gell-Mann scheme,[1] many authors[2)~7)] have investigated the characteristic isotopic spin changes in the decay processes of the strange particles, and pointed out the importance of the selection rule $|\Delta I|=1/2$. In fact, the experimental value[8] of the branching ratio between the two modes of the Λ^0-decay is consistent with the prediction of the $|\Delta I|=1/2$ rule. Moreover, the predictions of this rule are not inconsistent also with the data[9] for the branching ratio of the Σ^{+}-decay and the ratio of the lifetimes of Σ^{-} and Σ^{+}, provided that the parity does not conserve in these decays.[3)~5)] The validity of the latter assumption is evident nowadays.[10)~11)] However, it should be noted that these facts do not necessarily mean the non-existence of the other types of the decay interactions (e.g. $|\Delta I|=3/2$). For example, even under the postulate of time-reversal invariance (T-invariance), the theory basing on the $|\Delta I|=1/2$ rule involves three adjustable parameters for the case of Σ-decay (see later). Therefore, for the more definite test on this rule, it is necessary also to use the data for the other quantities. The situation is similar also for the case of Λ^0-decay. We shall mention in § 3. B on the possible method for such a detailed test on the $|\Delta I|=1/2$ theory.

Although the theoretical investigations which have thus far been carried out on the hyperon decays are almost confined within the test for the $|\Delta I|=1/2$ rule, we cannot be nowadays satisfied by only such a "test", itself. The reason is as follows: For the decays of K-mesons, it has already been evident that the selection rule $|\Delta I|=1/2$ is never absolute, and the contribution of the interactions other than the one of the type $|\Delta I|=1/2$ may amount to about 10% (in the decay amplitude) of the latter.[12] This fact suggests the possibility that the situation might be similar also in the hyperon decays.

Therefore, what we want to know is not only whether the $\Delta I = 1/2$ rule is consistent with the experimental data, but also how much contribution is given by each of the $|\Delta I| = 1/2$ and the other interactions (if they exist). Also, it is desired to obtain the quantitative information for other properties of the decay interactions, e.g. the ratio of the contributions from the parity-conserving and -reversing interactions.

One way to clarify the structure of the particles and their interactions is the "dynamical" method, with which one starts from the suitable models for the dynamical nature of the interactions and compares the computational results with the experimental data. For example, the attempt to extrapolate the "one-to-one" law[1)-3)] also for the case of the hyperon decay belongs to this method. However, here we want to emphasize also the importance of the kinematical attack. The latter needs only a few assumptions that are strongly confirmed by many experiences, and therefore, the conclusions deduced by this method are highly reliable. For example, it does not generally need the perturbational treatment of the effects of the strong interactions, in contrast with the "dynamical" method. Therefore, it would be very important to investigate the various possibilities involved in the kinematical analyses of the decay processes, and, if possible, to test the dynamical models thus far proposed for the nature of the interactions. This would also be helpful for the construction of a more reliable model. Of course, the kinematical method can generally give only limited information. However, at the present stage of the research on the strange particle decays, it may be said that even those limited possibilities are not yet investigated enough.

From such a view-point, we shall investigate in the present paper what information could be obtained by the kinematical analysis of the $(N+\pi)$ decays of hyperons, and what data are desired for obtaining such information. We shall also show the possibility of the experimental test for some dynamical models or assumptions thus far proposed on the nature of the interactions. Throughout the present paper, we postulate the following three assumptions:

1) Conservation of the total angular momentum in all reactions.

2) Isotopic spin assignments of the Nishijima and Gell-Mann scheme.

3) Conservation of parity and charge-independence in the strong reactions.

In addition to these, taking into account many experimental information, we treat here Λ and Σ as the spin 1/2 particles.[4)] However, the discussions in the present paper can be easily extended also to the cases of the spins of the hyperons not to be 1/2.

§ 2. Kinematics of the decays of charged Σ

Mainly for the illustration of the meanings of the notations, we first begin with a brief summary of the kinematics of the decays of the charged Σ-particles.

Observed decays of the Σ-particles into $(N+\pi)$ systems are

[1)] This terminology is used here to mean the assumption that the form of the ordinary space part of the decay interaction Hamiltonian is (symbolically) of the type $(1+\gamma_5)\cdot O_i$. See also § 3. *D*.

$$\begin{cases} \Sigma^+ \to p+\pi^0 & (0) \\ \Sigma^+ \to n+\pi^+ & (+) \\ \Sigma^- \to n+\pi^- & (-). \end{cases} \qquad (2\cdot1)$$

In the following, we call these three modes 0), +), and −), respectively, and distinguish the quantities related to each of these modes by the suffix i ($i=0,\ +,\ -$). The decay matrix element for each mode of (2·1) is given by

$$M_i=A_i+B_i \quad (i=0,\ +,\ -), \qquad (2\cdot2)$$

where A and B correspond to the two possible orbital angular momentum states $l=0$ (S) and $l=1$ (P), respectively, of the final $(N+\pi)$ system.

Since the isotopic spin of Σ-particle is 1, the only decay interactions that can contribute to the processes (2·1) are of the types $|\Delta I|=1/2,\ 3/2$, and $5/2$. Here the interaction of the type $|\Delta I|=1$, 2, for example, means that the interaction Hamiltonian representing it transforms like one component of a spinor under the rotation in the isotopic spin space. With the use of the Clebsch-Gordan coefficients, A's of (2·2) can be written as the sums of the contributions from the $|\Delta I|=1/2,\ 3/2$, and $5/2$ interactions:

$$\begin{cases} A_0=(1/3\sqrt{5})\,(\sqrt{10}\,a_3-4b_3+\sqrt{6}\,c_3)-(1/3\sqrt{2})\,(2a_1+b_1) \\ A_+=(1/3\sqrt{10})\,(\sqrt{10}\,a_3-4b_3+\sqrt{6}\,c_3)+(1/3)\,(2a_1+b_1) \\ A_-=a_3+\sqrt{(2/5)}\,b_3+\sqrt{(1/15)}\,c_3, \end{cases} \qquad (2\cdot3)$$

where a, b, and c represent the parts caused by the $|\Delta I|=1/2,\ 3/2$, and $5/2$ interactions, respectively, and the suffices 1 and 3 indicate the isotopic spin state of the final $(N+\pi)$ system. For example, a_3 is the "amplitude" for the transition into the ($I=3/2$, and $l=0$) state of $(N+\pi)$ system caused by the $|\Delta I|=1/2$ interaction. Corresponding to (2·3), the P-state parts of the decay matrix elements are given by

$$\begin{cases} B_0=(1/3\sqrt{5})\,(\sqrt{10}\,a_3'-4b_3'+\sqrt{6}\,c_3')-(1/3\sqrt{2})\,(2a_1'+b_1') \\ B_+=(1/3\sqrt{10})\,(\sqrt{10}\,a_3'-4b_3'+\sqrt{6}\,c_3')+(1/3)\,(2a_1'+b_1') \\ B_-=a_3'+\sqrt{(2/5)}\,b_3'+\sqrt{(1/15)}\,c_3'. \end{cases} \qquad (2\cdot4)$$

In the processes (2·1), the $|\Delta I|=5/2$ interaction can give rise to the final $I=3/2$ state only, and thus c_1 and c_1' are absent in the expressions (2·3) and (2·4).

By the unitarity of the S-matrix, the following relations hold between the phases of the "amplitudes" and the π-N scattering phase shifts δ's at the Σ-decay energy:

$$\begin{cases} a_j=\varepsilon_j^{(1)}|a_j|\cdot\exp i(\delta_j+\eta_{s1}) \\ b_j=\varepsilon_j^{(3)}|b_j|\cdot\exp i(\delta_j+\eta_{s3}) \\ c_3=\varepsilon_3^{(5)}|c_3|\cdot\exp i(\delta_3+\eta_{s5}) \\ a_j'=\varepsilon_j^{(1)\prime}|a_j'|\cdot\exp i(\delta_{j1}+\eta_{p1}) \end{cases}$$

$$\begin{cases} b_j' = \mathcal{E}_j^{(3)\prime} |b_j'| \cdot \exp i(\delta_{j1} + \eta_{p3}) \\ c_3' = \mathcal{E}_3^{(5)\prime} |c_3'| \cdot \exp i(\delta_{31} + \eta_{p5}), \quad (j = 1, 3) \end{cases} \tag{2·5}$$

where $\mathcal{E}$'s are the undetermined sign factors, and the upper suffices (1, 3, 5) of $\mathcal{E}$'s and the second suffices (1, 3, 5) of η's correspond to the $|\Delta I| = 1/2$, $3/2$, and $5/2$ interactions, respectively. Not η's themselves, but the differences between them are physically significant, which indicate the degree of violation of the time-reversal invariance in the decay processes in question. If the processes are T-invariant, then all the differences of η's vanish.

Now, let us pay notice to the following characteristic quantities of the decay processes:

1) The branching ratio ξ between the two modes ($i = 0$ and $+$) of Σ^+-decay, and the ratio ζ of the lifetimes of Σ^- and Σ^+,

$$\begin{cases} \xi \equiv w(\Sigma^+ \to p + \pi^0) / w(\Sigma^+ \to n + \pi^+) = (|A_0|^2 + |B_0|^2) / (|A_+|^2 + |B_+|^2), \\ \zeta \equiv \tau(\Sigma^-) / \tau(\Sigma^+) = (|A_0|^2 + |B_0|^2 + |A_+|^2 + |B_+|^2) / (|A_-|^2 + |B_-|^2). \end{cases} \tag{2·6}$$

2) The quantities α, β and γ (or a and ϕ), which are defined for each mode by

$$\begin{cases} \alpha = 2\,\mathrm{Re}(A^* B) / (|A|^2 + |B|^2), \\ \beta = 2\,\mathrm{Im}(A^* B) / (|A|^2 + |B|^2) \equiv (1 - \alpha^2)^{1/2} \cos\phi, \\ \gamma = (|A|^2 - |B|^2) / (|A|^2 + |B|^2) \equiv (1 - \alpha^2)^{1/2} \sin\phi, \\ (\alpha^2 + \beta^2 + \gamma^2 = 1). \end{cases} \tag{2·7}$$

The quantities (2·7) are the ones introduced by Lee and Yang,[1] and especially α's are the so-called asymmetry parameters.

Taking into account (2·3) ~ (2·5), all of the quantities defined by (2·6) and (2·7) can be easily expressed in terms of a's, etc. We shall first give those expressions for a simple case of the $|\Delta I| = 1/2$ interaction only (Subsection *A*), and next consider more general cases.

A. The case of only the $|\Delta I| = 1/2$ interaction

We first introduce the following parameters*:

$$\begin{cases} x_1 = \mathcal{E}_1^{(1)\prime} \mathcal{E}_3^{(1)\prime} \left[(|a_1|^2 + |a_1'|^2) / (|a_3|^2 + |a_3'|^2) \right]^{1/2}, \\ \tan\theta_1 = \mathcal{E}_1^{(1)\prime} \mathcal{E}_1^{(1)} |a_1| / |a_1'|, \quad \tan\phi_1 = \mathcal{E}_3^{(1)\prime} \mathcal{E}_3^{(1)} |a_3| / |a_3'|, \\ \eta_1 = \eta_{p1} - \eta_{s1}. \end{cases} \tag{2·8}$$

x_1 gives a measure for the ratio of the effective "amplitudes" of the final $I = 1/2$ and $3/2$ states caused by the $|\Delta I| = 1/2$ interaction. Similarly, θ_1 and ϕ_1 give that of the final S- and P-states.

In terms of these four parameters, the quantities defined by (2·6) and (2·7) are expressed as follows:

* These are quite similar to the parameters introduced by Kawaguchi.[3] θ_1 and ϕ_1 are equal to his θ and ϕ, respectively. However, x_1 is $(1/\sqrt{2})$ times of his x, and η_1 is absent in ref. 3).

$$\left.\begin{aligned}
&\xi=(2+2x_1^2-4A^+x_1)/(1+4x_1^2+4A^+x_1)\\
&\zeta=(1+2x_1^2)/3\\
&\alpha_0=4(Bx_1^2-Cx_1+D)/(\text{the numerator of }\xi)\\
&\gamma_0=-2[x_1^2\cos(2\theta_1)+2A^-x_1+\cos(2\Phi_1)]/(\text{the numerator of }\xi)\\
&\alpha_+=2(4Bx_1^2+2Cx_1+D)/(\text{the denominator of }\xi)\\
&\gamma_+=-[4x_1^2\cos(2\theta_1)-4A^-x_1+\cos(2\Phi_1)]/(\text{the denominator of }\xi)\\
&\alpha_-=2D\quad\gamma_-=-\cos(2\Phi_1),
\end{aligned}\right\}\qquad(2\cdot9)$$

where

$$\left.\begin{aligned}
&A^{\pm}=\sin\theta_1\sin\Phi_1\cos(\delta_1-\delta_3)\pm\cos\theta_1\cos\Phi_1\cos(\delta_{11}-\delta_{31})\\
&B=\sin\theta_1\cos\theta_1\cos(\delta_{11}-\delta_1+\eta_1)\\
&C=\sin\theta_1\cos\Phi_1\cos(\delta_{31}-\delta_1+\eta_1)+\cos\theta_1\sin\Phi_1\cos(\delta_{11}-\delta_3+\eta_1)\\
&D=\sin\Phi_1\cos\Phi_1\cos(\delta_{31}-\delta_3+\eta_1).
\end{aligned}\right\}\qquad(2\cdot10)$$

The expressions for β's are obtained from those of α's only by replacing the cos-functions of δ's and η's by the corresponding sin-functions. What is important here is that all of these expressions involve only four adjustable parameters defined by (2·8).

B. General case including all the types of interactions

We note first that, also in the case of the $|\Delta I|=3/2$ interaction only, the quantities (2·6) and (2·7) can be expressed in terms of only four adjustable parameters corresponding to (2·8), i.e.

$$\left.\begin{aligned}
&x_3=\mathcal{E}_1^{(3)\prime}\mathcal{E}_3^{(3)\prime}[(|b_1|^2+|b_1'|^2)/(|b_3|^2+|b_3'|^2)]^{1/2},\\
&\tan\theta_3=\mathcal{E}_1^{(3)\prime}\mathcal{E}_1^{(3)}|b_1|/|b_1'|,\quad\tan\Phi_3=\mathcal{E}_3^{(3)\prime}\mathcal{E}_3^{(3)}|b_3|/|b_3'|,\\
&\eta_3=\eta_{p3}-\eta_{s3}.
\end{aligned}\right\}\qquad(2\cdot11)$$

In the case of the $|\Delta I|=5/2$ interaction only, we need only two parameters Φ_5 and η_5, because of the absence of c_1 and c_1' in our formulas:

$$\tan\Phi_5=\mathcal{E}_3^{(5)\prime}\mathcal{E}_3^{(5)}|c_3|/|c_3'|,\quad\eta_5=\eta_{p5}-\eta_{s5}.\qquad(2\cdot12)$$

Finally, when there coexist the two types of the interactions, for example, $|\Delta I|=1/2$ and $3/2$, we need two more parameters defined by

$$y=\mathcal{E}_3^{(1)\prime}\mathcal{E}_3^{(3)\prime}[(|b_3|^2+|b_3'|^2)/(|a_3|^2+|a_3'|^2)]^{1/2},\quad\eta_{31}=\eta_{s3}-\eta_{s1},\qquad(2\cdot13)$$

in addition to the eight parameters defined by (2·8) and (2·11). Roughly speaking, y represents the ratio of the effective strengths of the parts of the $|\Delta I|=1/2$ and $3/2$ interactions which give rise to the final $I=3/2$ state, and η_{31} is the parameter relating to the phase difference between the S-state amplitudes caused by these two interactions. Similarly, in the case of co-existence of the $|\Delta I|=1/2$ and $5/2$ interactions, we need

two more parameters z and η_{51}, in addition to the six parameters defined by (2·8) and (2·12) :

$$z=\xi_3^{(1)\prime}\xi_3^{(5)\prime}\left[\left(|c_3|^2+|c_3'|^2\right)/\left(|a_3|^2+|a_3'|^2\right)\right]^{1/2},\quad \eta_{51}=\eta_{35}-\eta_{31}. \tag{2·14}$$

Thus, in the most general case including all of the above three types of interactions, we need altogether 14 independently adjustable parameters :

$$x_1,\ \theta_1,\ \Phi_1,\ \eta_1;\ x_3,\ \theta_3,\ \Phi_3,\ \eta_3;\ \Phi_5,\ \eta_5;\ y,\ z,\ \eta_{31},\ \text{and}\ \eta_{51}.$$

In any case, the expressions for $\tilde{\xi}$, ζ, etc., in terms of the relevant adjustable parameters can be obtained by straightforward (but rather cumbersome) calculations. However, since those expressions are generally very complicated, we shall not here give them explicitly. The details of those expressions for the most general case (with and without the postulate of T-invariance) will be given in Appendix.

§ 3. Possible experimental information on the Σ-decay interactions

A. General considerations

Among the quantities defined by (2·6) and (2·7), the number of the independently variable ones is only eight. For example, we can take

$$\begin{cases}\alpha \text{ and } \phi \text{ for each of three modes } (i=0,\ +,\ -),\\ \tilde{\xi} \text{ and } \zeta\end{cases} \tag{3·1}$$

as those independent quantities. This number, 8, indicates the upper limit of the possible experimental knowledge on the structure of the Σ-decay interaction. This is because that the physical properties of each mode $(i=0,\ +,\ -)$ of the decay processes are completely determined by the totally three quantities, i.e. the magnitudes of the two parts A_i and B_i of the decay matrix element M_i, and their relative phase. Among these totally 9 quantities, the absolute value of $|A_i|^2+|B_i|^2$ is arbitrarily adjustable within the numerical factor common to all of three modes by a suitable choice of the average strength of coupling. Thus, not the absolute values of A's and B's, but their relative magnitudes can serve for our purpose.

The above number, 8, is much smaller than the number, 14, of the independent parameters in the most general theory.* Therefore, even if all of the eight quantities (3·1) were measured with sufficient accuracy, it is impossible to determine the values of all these parameters and obtain the definite information on the details of the Σ-decay interactions. To obtain some information on these interactions from the experimental data, it is necessary to reduce the number of the independent parameters by the suitable assumptions or models on the nature of the interactions. For example, the postulate of

* This is mainly owing to that the possible $(N+\pi)$-decay of the Σ^0-particle can be hardly observed by the occurrence of the faster reaction, $\Sigma^0\to\Lambda_0+\gamma$. If the decay $\Sigma^0\to N+\pi$ were observable, then we had possessed six more data, i.e. $\tau(\Sigma^-)$ $\tau(\Sigma^0)$, the branching ratio between the two modes $\Sigma^0\to p+\pi^-$ and $\Sigma^0\to n+\pi^0$, and the quantities α and ϕ for each of these two modes.

T-invariance imposes that all γ_j's should vanish. Thus, there still remain the possibilities to determine the values of the parameters under such suitable assumptions, and in some cases, even to test those underlying assumptions by examining the consistency with the experimental data. There would exist many varieties of such attempts. In the following subsections, however, we shall consider only several examples for such possibilities that seem to be particularly interesting.

Before entering into the discussions of those examples, let us consider here which of the quantities (2·6) and (2·7) seem to be practically measurable. We have already known the following data[9),11)]:

$$\bar{\xi}=0.88\pm0.12,\quad \zeta=2.2\pm0.5, \tag{3·2}$$

and the "up-down" asymmetries,

$$P^+\cdot\alpha_0=-0.37\pm0.19,\quad P^+\cdot\alpha_+=-0.36\pm0.21,\quad P^-\cdot\alpha_-=-0.13\pm0.26, \tag{3·3}$$

where P^+ and P^- are the average polarization degrees of the parent Σ^+ and Σ^-, respectively. Since the signs and the magnitudes of P's are unknown, the data which are available at present are only those for

$$\bar{\xi},\ \zeta,\ \text{and}\ \alpha_0/\alpha_+. \tag{3·4}$$

However, some of the other quantities are also measurable in principle. For example, the ratio $\alpha_0:\alpha_+:\alpha_-$ would be obtained by the measurement of the ratio $P\cdot\alpha_0:P\cdot\alpha_+:P\cdot\alpha_-$ of the "up-down" asymmetries in the decays of $\Sigma^\pm$ which are produced by the reactions[17)]

$$\pi^++p\to\Sigma^++K^+\ \text{and}\ \pi^-+n\to\Sigma^-+K^0, \tag{3·5}$$

because these $\Sigma^\pm$ have the same degrees of polarization, provided that they are produced under the same experimental conditions (energy of the incident pion beam and the production angle). (Note that we have assumed the charge-independence of the strong reactions.) Moreover, Lee and Yang[16)] have recently suggested the method for directly obtaining the values of α and ϕ.* This is, in principle, not impossible at least for the

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* The value of α can be obtained by the measurement of the polarization (longitudinal) of the decay nucleon from the unpolarized Σ. The values of β and γ, or ϕ, are to be known from the transverse polarization of the decay nucleon from the completely polarized hyperon. However, the value of ϕ can be known (in principle) also in the practical case where the parent hyperon is partially polarized. In such a case, the polarization vector of the decay nucleon is given by

$$(1+P\cdot\alpha\cos\chi)^{-1}\,[(-\alpha-P\cos\chi)\,p+P\beta p\times s+P\gamma(p\times s)\times p],$$

instead of the formula given in ref. 16). Here, s and p are the unit vectors along the directions of the hyperon polarization and the nucleon emission, respectively. P is the polarization degree of the parent hyperon and χ is the angle between the directions of the pion emission and s ($\cos\chi=-p\cdot s$). In addition to this, note that P can be known from the values of α and the "up-down" asymmetry $P\cdot\alpha$. The value of P would be useful for the analysis of the production process of the hyperon. The direct measurement of α is thus very much desired, since it serves for the analysis not only of the decay but also of the production processes.

case of the decay nucleon being a proton (mode $i=0$). If α_0 is known, then also α_+ and α_- could be known through the knowledge on the ratio $\alpha_0:\alpha_+:\alpha_-$ obtained by the above-mentioned method. After all, we can conclude as follows: Under the favorable circumstances, it would be possible to measure the six quantities,

$$\bar{\xi},\ \zeta,\ \alpha_0,\ \alpha_+,\ \alpha_-,\ \text{and}\ \phi_0. \tag{3·6}$$

Even if we take into account that the direct measurement of α is considerably difficult owing to the very low energy of the decay nucleon, it would not be difficult to measure the ratios α_0/α_+ and α_0/α_-. Thus, the data at least for the four quantities,

$$\bar{\xi},\ \zeta,\ \alpha_0/\alpha_+,\ \text{and}\ \alpha_0/\alpha_- \tag{3·7}$$

would be expected to become available enough.

B. The $|\Delta I|=1/2$ theory

We shall first investigate the selection rule $\Delta I = 1/2$, as an example of the possibilities mentioned in *A*. As has already been pointed out, a rather large possibility is expected that the $|\Delta I|=1/2$ rule might be violated to some extent also in the case of the hyperon decay. However, on the other hand, another possibility has not yet been excluded that this rule is absolutely true for the case of the hyperon decay, and this might really be the characteristic difference between the decays of the hyperons and of the *K*-mesons. Whichever it may be, the important fact is that the evidence for the violation of the $|\Delta I|=1/2$ rule in the hyperon decay has not yet been found.[5] Therefore, among all, the detailed test on this rule would be the first problem which we must investigate.

The test on the *T*-invariant $|\Delta I|=1/2$ theory seems to be relatively simple, since this theory contains only three adjustable parameters x_1, θ_1, and ψ_1. The experimental knowledge on the values of the four quantities, e.g. $\bar{\xi}$, ζ, α_0/α_+ and α_0/α_-, or $\bar{\xi}$, ζ, α_0 and α_+, would overdetermine the values of those three parameters, and thus provide a detailed test for the theory.[7] Thus, the measurement of α_0 itself, or at least the ratio $\alpha_0:\alpha_+:\alpha_-$ by the method explained in *A* is very much desired.

In connection with this, note that it is sufficient for the determination of the values of those adjustable parameters to make use of the three known data for $\bar{\xi}$, ζ, and α_0, α_-. The parameter values thus determined give the individual values of α's. Or otherwise, in this case, we can express α's in terms of the three known quantities $\bar{\xi}$, ζ, and α_0, α_- by eliminating the parameters x_1, θ_1, and ψ_1 from the (T-invariant $\Delta I = 1/2$) theoretical expressions for $\bar{\xi}$, ζ and α's.[6,7] By such a method, Eguchi and Nagata[7] have obtained*

$$\alpha_0 = \pm\,(0.68 \pm 0.29),$$
$$\alpha_+ = \pm\,(0.70 \pm 0.28),$$
$$\alpha_- = \pm\,(0.84 \pm 0.23).$$

* As a means of testing the theory, they have considered the criterion that the magnitudes $P^\pm$ of the $\Sigma^\pm$-polarization deduced from these values of α's and the experimental data (3·3) should not be larger than unity. The above values of α's do not contradict this criterion.

Moreover, we want here to add the following remarks:

1) Also the general $|\Delta \boldsymbol{I}|=1/2$ theory without T-invariance postulate can be tested, if at least the five quantities ξ, ζ, α_0, α_+, and α_- are experimentally known by the method explained in *A*. This is owing to that the general $|\Delta \boldsymbol{I}|=1/2$ theory contains only four adjustable parameters, x_1, θ_1, Φ_1, and η_1.

2) Of course, to obtain the decisive conclusion on the validity of the $|\Delta \boldsymbol{I}|=1/2$ rule, it is best to start from the more general theory including all of the $|\Delta \boldsymbol{I}|=1/2$, $3/2$, and $5/2$ interactions, and examine whether the magnitudes y, yx_3, and z of the contributions of the $|\Delta \boldsymbol{I}|=3/2$ and $5/2$ interactions vanish or not. This is possible under the suitable assumptions on the nature of the interactions (see Subsection *D*). This is important not only because of the above-mentioned reason but also for that, when the $|\Delta \boldsymbol{I}|=1/2$ rule breaks down, the values of the above parameters are necessary as the important information on the Σ'-decay interactions.

C. The Postulate of T-invariance

As has been pointed out in refs. 10) and 16), when T-invariance holds, the parameters η_i's vanish exactly and thus all of $_i\beta$'s become very small. This is seen from the formulas (A·5) given in Appendix. Thus, if the measurements yield the large value for any one of $_i\beta$'s, then we can conclude that the postulate of T-invariance is surely violated. Note that this conclusion does not depend on any other assumptions (e. g. the $|\Delta \boldsymbol{I}|=1/2$ rule, etc.).

In addition to this, we want to point out the following facts: According to the formulas given in Appendix, the only changes, caused by the violation of T-invariance, in the expressions for ξ, ζ, α's, and γ's are the replacements,

$$\cos(\delta\text{'s}) \rightarrow \cos(\delta\text{'s}+\eta\text{'s}). \tag{3·8}$$

In practice, when T-invariance holds, the cos-functions of the left-hand side of (3·8) can be approximated by unity. This is owing to the very small values of the $\pi-N$ scattering phase shifts δ's at the Σ'-decay energy:

$$\begin{cases} \delta_1 \approx 13^0, \quad \delta_3 \approx -8^0, \quad \delta_{31} \approx \delta_{11} \approx -2.5^0, \text{ and thus} \\ \cos(\delta_1-\delta_3) \approx 0.93, \quad \cos(\delta_{11}-\delta_{31}) \approx 1.00, \\ \cos(\delta_{31}-\delta_3) \approx \cos(\delta_{11}-\delta_3) \approx 0.99, \\ \cos(\delta_{31}-\delta_1) \approx \cos(\delta_{11}-\delta_1) \approx 0.96. \end{cases} \tag{3·9}$$

For such small angles, the changes of cos-function caused by the relatively small changes of the angle are almost negligible, while the changes of the corresponding sin-function are proportional to those of the angle. Therefore, we can say roughly that the relatively small deviation from T-invariance can affect only the values of $_i\beta$'s. Unless the definite evidence for the violation of T-invariance is found, there would be a considerably high reliability in the values of the parameters which are determined consistently from the data for ξ, ζ, α's and γ's, with the postulate of T-invariance. Particularly, if the main decay interaction is the $|\Delta \boldsymbol{I}|=1/2$ type, then the expressions for ξ, ζ, and γ's are practically

independent of the T-invariance postulate. On the basis of the above considerations and because of the fact that any evidence against T-invariance has not yet been found, we shall assume this in the following.

Strictly speaking, even if the experimental data for all of ξ's are quite small, we cannot conclude that T-invariance holds, unless the values of all ζ's are found to be exactly zero. However, as has already been seen, the number of the parameters in the most general theory without T-invariance is too large to determine their values by the experiments. The situation is not much altered even under the postulate of T-invariance. (The number of the parameters in this case is 9.) Thus, as far as it concerns only with the decays (2·1), we can conclude as follows: Even if all of the quantities (3·1) were measured with sufficient accuracy, the best we can do is to examine the consistency of the T-invariant theory* with the suitable assumptions on the dynamical nature of the interactions.

D. Assumption of the decay Hamiltonian and the "one-to-one" law

The quantities $|a|$'s, etc., can generally involve not only the effects of the elementary decay interactions but also those of the strong (e. g. the final state π-N) interactions, and accordingly the relation between our parameters and the form of the decay Hamiltonian is very complicated. However, if the effects of the strong interactions can be neglected, then there holds some direct and simple relation between them. According to the results of the perturbational computation, those effects of the strong interactions are, in fact, very small. Of course, the perturbational treatment of the strong interactions is very doubtful in many respects. However, here we shall tentatively adopt this assumption.**

In addition to this, let us assume that both parts of the $|\Delta I|=1/2$ interaction which give rise respectively to the final $I=1/2$ and $3/2$ states have the same types of coupling. Here the "type of coupling" means the form of the spin- and momentum-dependence of the interaction Hamiltonian. For example, if the interaction is of the Yukawa type, then the Hamiltonian can be written symbolically as, say,

$$\psi\bar{\psi}_N(1+r\gamma_5)o_i\psi_N\cdot o_\pi,$$

except for the isotopic spin factors. The "type of coupling" means the form of the operator $(1+r\gamma_5)o_i$. Within a general frame of the terminology "$\Delta I=1/2$ interaction", the above two parts can have different types of coupling. However, in the usual dynamical approach, one would tentatively assume such a relatively simple Hamiltonian (e. g. see (3·23) or refs. 7) and 18)). Similarly, let us assume the corresponding two parts of the $|\Delta I|=3/2$ interaction to have the same types of coupling to each other. Under such assumptions, we have

$$\Theta_1=\Phi_1 \quad \text{and} \quad \Theta_3=\Phi_3. \tag{3·10}$$

* The only exception is the $|\Delta I|=1/2$ theory. We have seen in *B* that the test for this theory does not necessarily need T-invariance.

** Umezawa et al.[14] have insisted on the validity of this assumption, basing on their speculations on the difference of the effective domains of the strong and weak interactions.

As an alternative assumption, we can set also as

$$\Theta_1=\Theta_3\equiv\Theta,\quad \Phi_1=\Phi_3=\Phi_5\equiv\Phi, \tag{3·11}$$

instead of (3·10). This corresponds to the assumption that the types of coupling of the parts, responsible for the final $I=3/2$ state, of the $|\Delta\boldsymbol{I}|=1/2$, $3/2$ and $5/2$ interactions are of the same to each other, and the similar assumption for the parts responsible for the final $I=1/2$ state. (3·10) with the postulate of T-invariance reduces the number of parameters to only 7. (As for the case of (3·11), see later.)

Moreover, for example, if the part of the $|\Delta\boldsymbol{I}|=1/2$ interaction which gives rise to the final $I=1/2$ state has the $(V-A)$ coupling of the Yukawa type described symbolically as

$$\bar{\psi}_N(g-g'\gamma_5)\gamma_\mu\psi_\Sigma\partial_\mu\phi_\pi, \tag{3·12}$$

then[7]

$$a_1/a_1'=(g/g')\left[(M-m)/(M+m)\right]\{\left[(M+m)^2-\mu^2\right]/\left[(M-m)^2-\mu^2\right]\}^{1/2}, \tag{3·13}$$

where M, m and μ are the masses of Σ, nucleon and pion, respectively. Under the "one-to-one" law ($g=\pm g'$), (3·13) yields

$$\Theta_1\approx\pm 50^0. \tag{3·14}$$

Similarly, if it is the $(S-P)$ coupling, then

$$a_1/a_1'=(g/g')\{\left[(M+m)^2-\mu^2\right]/\left[(M-m)^2-\mu^2\right]\}^{1/2}, \tag{3·15}$$

and, under the "one-to-one" law,

$$\Theta_1\approx\pm 85^0. \tag{3·16}$$

Thus, corresponding to the particular assumption for the coupling types of the various interactions, the values of Θ's and Φ's are determined uniquely, and the number of the free parameters becomes only 4, at most, i. e. x_1, x_3, y and z. Therefore, if four quantities, at most, say (3·7), be measured, we could know the strengths of the $|\Delta\boldsymbol{I}|=3/2$ and $5/2$ interactions required by the particular assumption on the types of coupling. (For example, see (3·21).) Of course, it is possible also to test such an assumption on the coupling type, when the five quantities $\tilde{\xi}$, ζ, α_0, α_+, and α_- can be experimentally known. (See also the remarks towards the end of this Subsection.)

With a similar assumption as above, Eguchi-Nagata[7] and Nakagawa-Umezawa[15] have investigated the compatibility of the $|\Delta\boldsymbol{I}|=1/2$ and the "one-to-one" laws, and obtained the negative answer. However, their result can deny neither one of these two laws, since both of them are never the ones beyond the merely tentative assumptions. In view of the success of the "one-to-one" law in the lepton processes, it would be very interesting to investigate this law in the more general frame including also the interactions other than the $|\Delta\boldsymbol{I}|=1/2$ one. The above consideration show that such an investigation is, in fact, possible.

Finally, we want to add the following remarks :

1) Assumption (3·11) seems to reduce the number of parameters to 6. However, actually, it reduces this number to only 4. Under this assumption, the expressions for ξ, ζ, etc., become as

$$\begin{aligned}
&\xi=(1/2)[(2A-B)^2+4(1-q)AB]/[(A+B)^2-2(1-q)AB]\\
&\zeta=(1/18C^2)[(2A-B)^2+2(A+B)^2]\\
&\alpha_0=(4A^2s-4ABp+B^2t)/[(2A-B)^2+4(1-q)AB]\\
&\alpha_+=(A^2s+2ABp+B^2t)/[(A+B)^2-2(1-q)AB]\\
&\alpha_-=s,
\end{aligned}\tag{3·17}$$

with

$$A=1-2\sqrt{2/5}y+\sqrt{3/5}z,\quad B=2x_1+yx_2,\quad C=1+\sqrt{2/5}y+\sqrt{1/15}z,\tag{3·18}$$

and

$$p=\sin(\theta+\Phi),\quad q=\cos(\theta-\Phi),\quad s=\sin(2\Phi),\quad t=\sin(2\theta).\tag{3·19}$$

γ's are obtained from the corresponding α's by replacing the sin-functions of θ and Φ by the minus of the corresponding cos-functions. Thus, in this case, the essential parameters are eventually only B/A, C/A, θ and Φ. Under the particular assumption for the types of coupling, we have only two adjustable parameters B/A and C/A, which can be determined by the known data (3·2) (e. g. see (3·21)). It is remarkable that, in such a case, we cannot know the individual values of x_1, x_2, y, and z. The meanings of the parameters B/A and C/A are obvious: C/A gives a measure for the degree of violation of the $|\Delta I|=1/2$ rule, since the condition $y=z=0$ imposes that $C/A=1$. Similarly, B/A gives a measure for the ratio of the effective strengths of the two parts of the total decay interaction which give rise respectively to the final $I=1/2$ and $3/2$ states.

2) Under the combined assumption of (3·10) and (3·11),

$$\xi=(1/2)[(2A-B)/(A+B)]^2,\quad \zeta=(1/18C^2)[(2A-B)^2+2(A+B)^2],$$
$$\alpha_0=\alpha_+=\alpha_-=\sin(2\theta),\quad \gamma_0=\gamma_+=\gamma_-=-\cos(2\theta).\tag{3·20}$$

Note that the relations $\alpha_0=\alpha_+=\alpha_-$ and $\gamma_0=\gamma_+=\gamma_-$ hold independently of the coupling type, and thus, this model can be directly tested, if the radio $\alpha_0:\alpha_-:\alpha$ be measured. The data (3·3) seem to indicate that $\alpha_0=\alpha_-$. If we extrapolate this also to α_- and assume (3·10) and (3·11) to be true, then the data (3·2) are quite inconsistent with the $|\Delta I|=1/2$ rule. Namely, the expressions (3·20) and the data (3·2) lead to

$$\begin{cases} B/A\approx 0.29\pm 0.04 \\ |C/A|\approx 0.40\pm 0.05 \end{cases}\quad\text{or}\quad \begin{cases} B/A\approx -(10.2\pm 2.8) \\ |C/A|\approx 2.8\pm 0.6. \end{cases}\tag{3·21}$$

Moreover, if the coupling is the $(V-A)$ or $(S-P)$ one of the Yukawa type with $g=\pm g'$ (the "one-to-one" law), then we get by (3·14) and (3·16)

$$\alpha_0=\alpha_+=\alpha_-=\pm 1,\quad\text{or}\quad \alpha_0=\alpha_+=\alpha_-=\pm 0.2,\tag{3·22}$$

respectively. Both of these two cases are not inconsistent with the data (3·3). It is obvious that the previously quoted works of refs. 7) and 18) are the special cases of the present consideration.

E. Conclusion

We have investigated what information could be obtained, at least in principle, by the measurements of the decay processes (2·1). The essential limitation of such information (owing to the insufficiency of the number of the observable quantities) is clarified, and within this limitation, the various possibilities are discussed. Especially, the measurement of the individual values of α's, or at least, that of the ratio $\alpha_0 : \alpha_+ : \alpha_-$, is very much desired for the more definite tests on the $|\Delta \boldsymbol{I}| = 1/2$ rule and the analysis discussed in *D*.

We have seen that, as long as the effects of the strong interactions are neglected, the values of our parameters θ's and Φ's have certain correspondences with the "dynamical" assumption for the form of the decay Hamiltonian. In other words, the "dynamical" approach, in which only the lowest order effects of the decay interactions are taken into account, is never the one beyond our "phenomenological" analysis. For example, to assume tentatively the decay Hamiltonian

$$H = [g_0 \bar{P}(1+\gamma_5)\gamma_\mu \Sigma^+ \partial_\mu \Pi^0 + g_+ \bar{N}(1+\gamma_5)\gamma_\mu \Sigma^+ \partial_\mu \Pi^- + g_- \bar{N}(1+\gamma_5)\gamma_\mu \Sigma^- \partial_\mu \Pi^+] + \text{h.c.} \tag{3·23}$$

is equivalent to our combined assumption (3·10), (3·11), and (3·14). In fact, except for the absolute values of g's which relate to the absolute probabilities of decays, the essentially free parameters involved in (3·23) are g_0/g_+, and g_0/g_-, whose number is only two in agreement with that of the parameters appearing in (3·20).

However, the analysis of the present paper is advantageous in that, in general, it does not necessarily need the neglect or the perturbational treatment of the strong interactions. For example, even if the full effect of the *final* state interactions are taken into account, the relation (3·11), and accordingly (3·17) would be expected to hold, provided that the types of coupling are same for all of the $|\Delta \boldsymbol{I}| = 1/2$, $3/2$, and $5/2$ interactions. Similarly, the discussion in *B* on the $|\Delta \boldsymbol{I}| = 1/2$ theory is completely independent of the assumption for the dynamical nature of the interactions. Thus, we feel that the analysis of the present paper covers almost all that we can do at present on the Σ-decays.

§ 4. Decay of the Λ^0-particle

A similar analysis to the one for the Σ-decays is applicable also to the case of the Λ^0-decay. In the following, we shall briefly mention on this case.

A. Kinematics

Since the isotopic spin of Λ^0 is zero, the types of the interactions which can contribute to the decay $\Lambda^0 \to N + \pi$ are only $|\Delta \boldsymbol{I}| = 1/2$ and $3/2$. The final states produced by these two types of the interactions are purely $I = 1/2$ and $3/2$, respectively, and thus

it is unnecessary to add the suffix (1 or 3) to the quantities a and b.* Thus, even in the most general theory, the free parameters are only

$$\begin{cases} \gamma=\mathcal{E}^{(1)\prime}\mathcal{E}^{(3)\prime}\left[(|b|^2+|b'|^2)(|a|^2+|a'|^2)\right]^{1/2}, \\ \tan\Theta=\mathcal{E}^{(1)\prime}\mathcal{E}^{(1)}\cdot|a/|a'|, \quad \tan\Phi=\mathcal{E}^{(3)\prime}\mathcal{E}^{(3)}\cdot|b|/|b'|, \\ \gamma_s=\gamma_{s3}-\gamma_{s1}, \quad \eta_p=\gamma_{p3}-\gamma_{p1}, \quad \eta=\gamma_{p1}-\gamma_{s1}, \end{cases} \tag{4·1}$$

where

$$\begin{cases} a=\mathcal{E}^{(1)}|a|\cdot\exp i(\delta_1+\gamma_{s1}), \quad a'=\mathcal{E}^{(1)\prime}a'\cdot\exp i(\delta_{11}+\gamma_{p1}), \\ b=\mathcal{E}^{(3)}|b|\cdot\exp i(\delta_3+\gamma_{s3}), \quad b'=\mathcal{E}^{(3)\prime}b'\cdot\exp i(\delta_{31}-\gamma_{p3}). \end{cases} \tag{4·2}$$

Θ, Φ, and γ in (4·1) correspond to Θ_1, Φ_3, and γ_1 for the case of Σ-decays, respectively.

We can express the branching ratio $\hat{\xi}_\Lambda\equiv w(\Lambda^0\to p+\pi^-)/w(\Lambda^0\to n+\pi^0)$, the asymmetry parameters α's and so on, in terms of the above six free parameters. Those expressions basing on the most general theory will be given in Appendix. (A·10). Under the postulate of T-invariance, they become as follows:

$$\begin{cases} \hat{\xi}_\Lambda=[2-2\sqrt{2}\,\gamma\cos(\Theta-\Phi)+\gamma^2]/[1+2\sqrt{2}\,\gamma\cos(\Theta-\Phi)+2\gamma^2], \\ \alpha_a=[2\sin(2\Theta)-2\sqrt{2}\,\gamma\sin(\Theta+\Phi)+\gamma^2\sin(2\Phi)]/(\text{numerator of }\hat{\xi}_\Lambda), \\ \alpha_b=[\sin(2\Theta)+2\sqrt{2}\,\gamma\sin(\Theta+\Phi)+2\gamma^2\sin(2\Phi)]/(\text{denominator of }\hat{\xi}_\Lambda), \\ \gamma_a=-[2\cos(2\Theta)-2\sqrt{2}\,\gamma\cos(\Theta+\Phi)+\gamma^2\cos(2\Phi)]/(\text{numerator of }\hat{\xi}_\Lambda), \\ \gamma_b=-[\cos(2\Theta)+2\sqrt{2}\,\gamma\cos(\Theta+\Phi)+2\gamma^2\cos(2\Phi)]/(\text{denominator of }\hat{\xi}_\Lambda), \\ \text{and}\quad \beta_a\approx\beta_b\approx0, \end{cases} \tag{4·3}$$

where the suffices a and b denote that the quantities are respectively corresponding to the decay modes,

$$\Lambda^0\to p+\pi^-(a), \text{ and } \Lambda^0\to n+\pi^0\ (b). \tag{4·4}$$

In (4·3), we have approximated the cos-functions of the differences of δ's by unity, since the $\pi-N$ scattering phase shifts are very small at the Λ^0-decay energy.

Similarly, under the $|\Delta I|=1/2$ theory,

$$\begin{cases} \hat{\xi}_\Lambda=2, \quad \gamma_a=\gamma_b=-\cos(2\Theta), \\ \alpha_a=\alpha_b=\sin(2\Theta)\cos(\delta_{11}-\delta_1+\eta)\approx\sin(2\Theta)\cos\eta, \\ \beta_a=\beta_b=\sin(2\Theta)\sin(\delta_{11}-\delta_1+\eta)\approx\sin(2\Theta)\sin\eta. \end{cases} \tag{4·5}$$

In (4·5), we have not postulated T-invariance.

* The meanings of the notations are similar as those of § 2 and 3. Although the interactions for the Λ^0-decay are not necessarily same as those for the Σ-decay, we use here the same letters a and b to avoid unnecessary complications.

B. Possible experimental information

Through similar discussions to those in the previous sections, what can help to determine the nature of the decay interactions of this case are at most five quantities, e. g. $\tilde{\xi}_\Lambda$, α_a, α_b, ϕ_a and ϕ_b. The data obtained thus far are[6),9)]

$$\begin{cases} w(\Lambda^0 \to n+\pi^0)/w(\Lambda^0, \text{ total}) = 0.32 \pm 0.05 \quad (\tilde{\xi}_\Lambda \approx 2.1) \\ \bar{P}\cdot\alpha_a = +0.44 \pm 0.11, \end{cases} \tag{4·6}$$

where $\bar{P}$ is the average degree of polarization of the parent Λ^0.

Under such a circumstance and with the formulas (4·3) ~ (4·5) and (A·10), we can draw several conclusions on the possible information obtainable from the experimental analysis of the Λ^0-decays. Since the details of the discussions are quite similar to those of § 3, we shall not here repeat them. However, we want here to emphasize the importance of the measurement of α_a and α_b, or at least their ratio* α_a/α_b. This is owing to the following reasons:

1) Analogously to the discussion in § 3. *B*, let us assume T-invariance and $\theta=\Phi$. In this case, (4·3) reduces to

$$\begin{cases} \tilde{\xi}_\Lambda = [(\sqrt{2}-\gamma)/(1+\sqrt{2}\,\gamma)]^2 \\ \alpha_a = \alpha_b = \sin(2\theta) \quad \gamma_a = \gamma_b = -\cos(2\theta). \end{cases} \tag{4·7}$$

On the other hand, we have seen that, if the $|\Delta \boldsymbol{I}|=1/2$ theory be true, the relations $\alpha_a=\alpha_b$ and $\phi_a=\phi_b$ hold independently of any other assumptions. Thus, the measurement of the ratio $\alpha_a:\alpha_b$ provides a means of test for the above two models. However, the following should be noted: From the conditions $\alpha_a=\alpha_b$ and $\phi_a=\phi_b$ alone, we cannot judge which of the above two models is true. When the measurement shows $\alpha_a=\alpha_b$, the test of the $|\Delta \boldsymbol{I}|=1/2$ rule requires a more accurate measurement of $\tilde{\xi}_\Lambda$. If we assume (4·7) to be true, then the data (4·6) confine the permissible region of the value of γ within

$$-2.5 \gtrsim \gamma \gtrsim -2.8 \quad \text{or} \quad +0.03 \gtrsim \gamma \gtrsim -0.03. \tag{4·8}$$

2) If the individual values of α_a and α_b were known, then we could determine the parameters of the general T-invariant theory including both of the $|\Delta \boldsymbol{I}|=1/2$ and 3/2 interactions. Moreover, if we neglect the effects of the strong interactions, then, from the values of θ and Φ thus determined, we would be able to know the coupling type of the decay interactions. For example, if we assume the coupling type of both of the $|\Delta \boldsymbol{I}|=1/2$ and 3/2 interactions to be $(V-A)$ or $(S-P)$ of the Yukawa type with the "one-to-one" law, then we get (similarly to (3·22))

$$\alpha_a = \alpha_b \approx \pm 0.9 \quad \text{or} \quad \pm 0.1, \tag{4·9}$$

* The measurement of at least the ratio α_a/α_b would certainly be possible, although it would be more difficult than the measurement of the corresponding ratio $\alpha_0:\alpha_+:\alpha_-$ in the Σ-decays, owing to that both of the two decay products, n and π^0, in the mode *b*) are neutral.

respectively. It is remarkable that the $(V-A)$ coupling is not inconsistent with the data $(4\cdot6)$, but the $(S-P)$ coupling is completely ruled out in this case. (Note that the magnitude of $\bar{P}$ in $(4\cdot6)$ cannot be larger than unity.)

Appendix. General formulas for ξ, ζ, etc.

A. The case of the Σ-decays

1) T-invariant theory including all the possible interactions

The expressions for ξ and ζ are given by

$$\xi=N_0/N_+ \quad \text{and} \quad \zeta=(N_0+N_-)/(9N_-), \tag{A·1}$$

where

$$\begin{aligned}
N_0=&2+2x_1^2-4x_1\cos(\theta_1-\phi_1)+(8/5)y^2\{2+(5/16)x_3^2+(5/\sqrt{10})x_3\\
&\times\cos(\theta_3-\phi_3)\}-2y\{4\sqrt{2/5}[\cos(\phi_1-\phi_3)-x_1\cos(\theta_1-\phi_3)]+x_3\\
&\times\cos(\theta_3-\phi_1)-x_1x_3\cos(\theta_1-\theta_3)\}+(6/5)z^2+2\sqrt{3/5}z\{2\cos(\phi_1-\phi_5)\\
&-2x_1\cos(\theta_1-\phi_5)-x_3y\cos(\theta_3-\phi_5)-4\sqrt{2/5}y\cos(\phi_3-\phi_5)\},\\
N_+=&1+4x_1^2+4x_1\cos(\theta_1-\phi_1)+(8/5)y^2\{1+(5/8)x_3^2-(5/\sqrt{10})x_3\\
&\times\cos(\theta_3-\phi_3)\}-2y\{2\sqrt{2/5}[\cos(\phi_1-\phi_3)+2x_1\cos(\theta_1-\phi_3)]-x_3\\
&\times\cos(\theta_3-\phi_1)-2x_1x_3\cos(\theta_1-\theta_3)\}+(3/5)z^2+2\sqrt{3/5}z\{\cos(\phi_1-\phi_5)\\
&+2x_1\cos(\theta_1-\phi_5)+x_3y\cos(\theta_3-\phi_5)-2\sqrt{2/5}y\cos(\phi_3-\phi_5)\},\\
N_-=&1+(2/5)y^2+2\sqrt{2/5}y\cos(\phi_1-\phi_3)+(1/15)z^2\\
&+(2/\sqrt{15})z\cos(\phi_1-\phi_5)+(2/5)\sqrt{2/3}yz\cos(\phi_3-\phi_5),
\end{aligned} \tag{A·2}$$

and thus the numerator of ζ is

$$\begin{aligned}
N_0+N_+=&3[1+2x_1^2+(8/5)y^2\{1+(5/16)x_3^2\}-4\sqrt{2/5}y\cos(\phi_1-\phi_3)\\
&+2yx_1x_3\cos(\theta_1-\theta_3)+(3/5)z^2+2\sqrt{3/5}z\cos(\phi_1-\phi_5)\\
&-(4\sqrt{6}/5)yz\cos(\phi_3-\phi_5)].
\end{aligned} \tag{A·3}$$

The expressions for α's are as follows:

$$\begin{aligned}
\alpha_0=&(2/N_0)[\sin(2\phi_1)-2x_1\sin(\theta_1+\phi_1)+x_1^2\sin(2\theta_1)\\
&-y\{4\sqrt{2/5}[\sin(\phi_1+\phi_3)-x_1\sin(\theta_1+\phi_3)]\\
&+x_3\sin(\phi_1+\theta_3)-x_1x_3\sin(\theta_1+\theta_3)\}\\
&+(4/5)y^2\{2\sin(2\phi_3)+(5/\sqrt{10})x_3\sin(\theta_3+\phi_3)+(5/16)x_3^2\sin(2\theta_3)\}\\
&+\sqrt{3/5}z\{2\sin(\phi_1+\phi_5)-2x_1\sin(\theta_1+\phi_5)-4\sqrt{2/5}y\sin(\phi_3+\phi_5)\\
&-yx_3\sin(\theta_3+\phi_5)\}+(3/5)z^2\sin(2\phi_5)],
\end{aligned}$$

$$\begin{aligned}\alpha_+ = (2/N_+)[&(1/2)\sin(2\Phi_1) + 2x_1\sin(\Theta_1+\Phi_1) + 2x_1^2\sin(2\Theta_1)\\ &-y\{2\sqrt{2/5}[\sin(\Phi_1+\Phi_3) + 2x_1\sin(\Theta_1+\Phi_3)]\\ &-x_3\sin(\Theta_3+\Phi_1) - 2x_1x_3\sin(\Theta_1+\Theta_3)\}\\ &+(4/5)y^2\{\sin(2\Phi_3) - (5/\sqrt{10})x_3\sin(\Theta_3+\Phi_3) + (5/8)x_3^2\sin(2\Theta_3)\}\\ &+\sqrt{3/5}\,z\{\sin(\Phi_1+\Phi_5) + 2x_1\sin(\Theta_1+\Phi_5) - 2\sqrt{2/5}\,y\sin(\Phi_3+\Phi_5)\\ &+yx_3\sin(\Theta_3+\Phi_5)\} + (3/10)z^2\sin(2\Phi_5)],\\ \alpha_- = (2/N_-)[&(1/2)\sin(2\Phi_1) + \sqrt{2/5}\,y\sin(\Phi_1+\Phi_3) + (1/5)y^2\sin(2\Phi_3)\\ &+z\{(1/\sqrt{15})\sin(\Phi_1+\Phi_5) + (1/5)\sqrt{2/3}\,y\sin(\Phi_3+\Phi_5)\}\\ &+(1/30)z^2\sin(2\Phi_5)],\end{aligned} \tag{A·4}$$

The expressions for γ_i's are obtained from those for α's by replacing the sin-functions of Θ's and Φ's by the minus of the corresponding cos-functions. Finally, the expressions for β_i's are obtained from those for the corresponding α's by the following replacements:

$$\begin{aligned}&\sin(\Phi_i+\Phi_j)\to 0.1\sin(\Phi_i+\Phi_j),\\ &\sin(\Theta_i+\Theta_j)\to -0.27\sin(\Theta_i+\Theta_j),\\ &\sin(\Phi_i+\Theta_j)\to 0.1\sin\Phi_i\cos\Theta_j - 0.27\sin\Theta_j\cos\Phi_i\\ &\qquad = -(1/2)[0.17\sin(\Phi_i+\Theta_j) - 0.37\sin(\Phi_i-\Theta_j)],\\ &\qquad (i,\ j=1,\ 3,\ 5).\end{aligned} \tag{A·5}$$

In (A·1) ~ (A·4), we have approximated the cosine functions of the differences of δ's [e. g. $\cos(\delta_1-\delta_3)$, etc.] by unity. This gives rise to the errors of only a few percent. The exact expressions (without this approximation) can be easily obtained from the expressions in the most general case only by setting all η_i's as zero. [See (A·7) ~ (A·9).] In (A·5), we have taken into account that, by (3·7),

$$\begin{cases}\sin(\delta_{31}-\delta_3)\approx\sin(\delta_{11}-\delta_3)\approx 0.1\\ \sin(\delta_{31}-\delta_1)\approx\sin(\delta_{11}-\delta_1)\approx -0.27.\end{cases} \tag{A·6}$$

2) The most general theory without the postulate of T-invariance.

In this case, the required expressions are obtained from the formulas (A·1) ~ (A·4) of the T-invariant theory by the following replacements:

a. The expression for ξ and ζ

$$N_i\to N_i' \quad (i=0,\ +,\ -), \tag{A·7}$$

where N_i' is the quantity obtained from N_i by the replacements,

$$\begin{aligned}\cos(\Theta_i-\Phi_j)\to&\sin\Theta_i\sin\Phi_j\cos(\delta_1-\delta_3+\eta_{ij})\\ &+\cos\Theta_i\cos\Phi_j\cos(\delta_{11}-\delta_{31}+\eta_i-\eta_j+\eta_{ij}),\\ \cos(\Phi_i-\Phi_j)\to&\sin\Phi_i\sin\Phi_j\cos\eta_{ij}+\cos\Phi_i\cos\Phi_j\cos(\eta_i-\eta_j+\eta_{ij}),\end{aligned}$$

$$\cos(\theta_i-\theta_j)\to\sin\theta_i\sin\theta_j\cos\eta_{ij}+\cos\theta_i\cos\theta_j\cos(\eta_i-\eta_j+\eta_{ij}),$$

with $\eta_{ij}=-\eta_{ji}$, $\eta_{ii}=0$, and $\eta_{35}=\eta_{31}-\eta_{51}$. $(i,\ j,=1,\ 3,\ 5)$ (A·8)

b. The expressions for α's and γ's

These require, in addition to (A·7), the following replacements for the sin-functions of θ's and Φ's in the expressions (A·4) of α's, and the corresponding cos-functions in the expressions of γ's:

$$\begin{aligned}
\sin(\Phi_i+\Phi_j)&\to\sin\Phi_i\cos\Phi_j\cos(\delta_{31}-\delta_3+\eta_j+\eta_{ji})\\
&\quad+\cos\Phi_i\sin\Phi_j\cos(\delta_{31}-\delta_3+\eta_i-\eta_{ji}),\\
\sin(\theta_i+\theta_j)&\to\sin\theta_i\cos\theta_j\cos(\delta_{11}-\delta_1+\eta_j+\eta_{ji})\\
&\quad+\cos\theta_i\sin\theta_j\cos(\delta_{11}-\delta_1+\eta_i-\eta_{ji}),\\
\sin(\theta_i+\Phi_j)&\to\sin\theta_i\cos\Phi_j\cos(\delta_{31}-\delta_1+\eta_j+\eta_{ji})\\
&\quad+\cos\theta_i\sin\Phi_j\cos(\delta_{11}-\delta_3+\eta_i-\eta_{ji})\\
\cos(\Phi_i+\Phi_j)&\to\cos\Phi_i\cos\Phi_j\cos(\eta_j-\eta_i+\eta_{ji})-\sin\Phi_i\sin\Phi_j\cos\eta_{ji},\\
\cos(\theta_i+\theta_j)&\to\cos\theta_i\cos\theta_j\cos(\eta_j-\eta_i+\eta_{ji})-\sin\theta_i\sin\theta_j\cos\eta_{ji},\\
\cos(\theta_i+\Phi_j)&\to\cos\theta_i\cos\Phi_j\cos(\delta_{11}-\delta_{31}+\eta_i-\eta_j+\eta_{ij})\\
&\quad-\sin\theta_i\sin\Phi_j\cos(\delta_1-\delta_3+\eta_{ij}).
\end{aligned}\tag{A·9}$$

Note that the replacement (A·9) keeps $\cos(2\Theta_i)$ and $\cos(2\Phi_i)$ unchanged.

c. The expressions for β's

These are obtained from the expressions for α's (in the present case) by replacing the cos-functions of δ's and η's by the corresponding sin-functions.

B. The case of the Λ^0-decays

$$\begin{aligned}
\hat{\xi}_\Lambda&=(2+y^2-2\sqrt{2}\,A^+\cdot y)/(1+2y^2+2\sqrt{2}\,A^+\cdot y),\\
\alpha_a&=2(2B+Dy^2-\sqrt{2}\,Cy)/(2+y^2-2\sqrt{2}\,A^+\cdot y),\\
\alpha_b&=2(B+2Dy^2+\sqrt{2}\,Cy)/(1+2y^2+2\sqrt{2}\,A^+\cdot y),\\
\gamma_a&=-[2\cos(2\Theta)+y^2\cos(2\Phi)+2\sqrt{2}\,A^-\cdot y]/(2+y^2-2\sqrt{2}\,A^+\cdot y)\\
\gamma_b&=-[\cos(2\Theta)+2y^2\cos(2\Phi)-2\sqrt{2}\,A^-\cdot y]/(1+2y^2+2\sqrt{2}\,A^+\cdot y),
\end{aligned}\tag{A·10}$$

where

$$\begin{aligned}
A^\pm&=\sin\Theta\sin\Phi\cos(\delta_3-\delta_1+\eta_s)\pm\cos\Theta\cos\Phi\cos(\delta_{31}-\delta_{11}+\eta_p),\\
B&=\sin\Theta\cos\Theta\cos(\delta_{11}-\delta_1+\eta),\\
C&=\sin\Theta\cos\Phi\cos(\delta_{31}-\delta_1+\eta_p+\eta)+\cos\Theta\sin\Phi\cos(\delta_{11}-\delta_3-\eta_s+\eta),\\
D&=\sin\Phi\cos\Phi\cos(\delta_{31}-\delta_3+\eta_p-\eta_s+\eta).
\end{aligned}\tag{A·11}$$

β's are obtained from α's by replacing the cos-functions of δ's and γ's by the corresponding sin-functions.

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Perturbational Calculations of Propagators of the Elementary Particles interacting with Gravitational Field

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Expanding Deser's propagators[1] by the coupling constant according to Laurent[2], we can calculate the propagators with the gravitational correction by the prescription of Hu[3]. If we use these corrected propagators in the calculations of S-matrices, each Feynman diagram converges except for the cases of vertex number $n=2$ corresponding to graviton self-energy due to boson and to graviton-graviton scattering, the former of which may be dropped because of gauge invariance. And the larger the degree of diagram is, the better the convergence is.

§ 1. Introduction and summary

Recently Deser[1] pointed out a possibility that the usual singularities of propagators may be removed by taking into account the interaction with the gravitational field. According to him the singularities would not appear because the measure of Riemannian space which produces the singularities becomes zero and, therefore, gives no contribution to Feynman's path integral[4]. But as Deser himself said, it is not easy to carry out the path-integration and a rigorous proof of his conjecture would seem to require a better developed mathematical theory of measure in functional integrations. At present we have no way, except for the perturbational calculation, to study the situations more concretely, but, as Deser said, the power series expansion by the coupling constant may not be right for high energies and, therefore, the power series expansion solution of Laurent[2] have maybe no meaning for this purpose because the theory contains the derivative coupling and diverges if we use his original forms of propagators. But as Hu[3] calculated the S-matrix in the pseudoscalar meson theory having the pseudovector coupling with nucleon, when we use the propagators corrected by the second-order self-energy parts, the situation is the same as in an electrodynamics even if there are derivative couplings and so the power series expansion by the coupling constant may have a meaning, though it is against Deser's intension. The Feynman method is known to be equivalent to the canonical one for the usual fields. When we expand Deser's propagators by the coupling constant according to Laurent's prescription and calculate them with Hu's method, we see that the propagators of fermion and boson* behave like those calculated by the usual perturbational method in the canonical formalism[5] for large momenta. Whether or not the two procedures differ somewhat in the gravitational field the Feynman method seems

* Here we use bosons as including no graviton.

to provide a consistent approach. It seems to avoid at least some of the difficulties encountered by extensions of the canonical method to the highly nonlinear problem in the gravitational field because in the Feynman method we need not consider all operators at once ; we may begin with constructing just one of them, or some other operator that we think might be simpler than the basic operators of the canonical theory[6], and this prescription has the further advantage of exhibiting crucial differences between matter and gravitational interactions ; i.e., the matter Lagrangian simply has no free-particle part any longer[1].

In Sec. 2 we expand Deser's propagators by the coupling constant. In Sec. 3 we calculate the second-order self-energy of fermion due to the gravitational field and obtain the propagator of fermion corrected by that self-energy part. We see that the corrected propagator behaves like p^{-3} for large momenta p. In Sec. 4 we calculate the second-order self-energy of boson due to gravitational field and that of graviton due to fermion field and obtain the propagators of boson and graviton corrected by that self-energy part. We see that the both corrected propagators behave like p^{-4} for large momenta p. In Sec. 5 we calculate an S-matrix with these corrected propagators and find that the term corresponding to each Feynman diagram converges except for the cases which have the vertex number $n=2$ and correspond to the graviton self-energy due to boson and to the graviton-graviton scattering, the former of which may be dropped because of gauge invariance. But in the perturbational calculations we must use the cut-off method to remove the divergences and, therefore, it is against Deser's intention. Accordingly if Deser's conjecture is right, the essential contributions to his theory must come from the strong interactions with the gravitational field or large metric tensors which cannot be treated perturbationally.

Finally there is the problem of new poles discussed by Feldman[7]. Here we can only say that at present there is no way except for his so-called " rule ".

§ 2. Power series expansion of Deser's propagators

The propagator of fermion interacting with the gravitational field is[1),2)]

$$\langle T(\psi(1)\bar{\psi}(1'))\rangle_0 = N^{-1}\int \psi(1)\bar{\psi}(1')\exp[-iI(\psi,\bar{\psi},\boldsymbol{g}_{\mu\nu})]\delta\psi\delta\bar{\psi}\delta\boldsymbol{g}_{\mu\nu}, \tag{1}$$

where T is Wick's ordering operator and the integral of right-hand side is Feynman's path-integral[4] and

$$I(\psi,\bar{\psi},\boldsymbol{g}_{\mu\nu}) = \int L d^4x, \tag{2}$$

$$L = \sqrt{g}\,\{\beta^{-2}R - i\bar{\psi}(\gamma^a\nabla_a - \mu)\psi\} \quad (\mu \text{ is a fermion mass}), \tag{3}$$

$$N = \int \exp[-iI(\psi,\bar{\psi},\boldsymbol{g}_{\mu\nu})]\delta\psi\delta\bar{\psi}\delta\boldsymbol{g}_{\mu\nu}, \tag{4}$$

$$\beta = \sqrt{2K} \quad (K \text{ is Einstein's gravitational const.}), \tag{5}$$

$$R=g^{\mu\nu}R_{\mu\nu},$$

$$\boldsymbol{g}_{\mu\nu}=\sqrt{g}\,g_{\mu\nu},$$

$$g=-\text{determinant}\ (g_{\mu\nu}).$$

$g^{\mu\nu}$ are metric tensors and $R_{\mu\nu}$ are Einstein's curvature tensors. Γ_α are covariant derivatives defined by

$$\Gamma_\mu\psi=(\partial_\mu-\Gamma_\mu)\psi,$$

$$\partial\gamma_\mu/\partial x^\nu=\gamma^\alpha\Gamma_{\alpha\mu\nu}+\Gamma_\nu\gamma_\mu-\gamma_\mu\Gamma_\nu,$$

where $\Gamma_{\alpha\mu\nu}$ are usual affine connections.

According to Laurent[2] if we assume that

$$\boldsymbol{g}^{\mu\nu}=\overset{\circ}{g}{}^{\mu\nu}-\beta\chi^{\mu\nu}, \tag{6}$$

($\overset{\circ}{g}{}^{\mu\nu}$ are metric tensors in the Minkowski space) and expand eq. (1) with respect to β we obtain

$$\langle T(\psi(1)\bar{\psi}(1'))\rangle_0=N^{-1}\int\psi(1)\bar{\psi}(1')\exp\Big[-i\int(L_D^{(1)}+L_G^{(1)}-L_{ie}^{(1)})\,d^4x\Big]\delta\psi\delta\bar{\psi}\delta\chi_{\mu\nu}, \tag{7}$$

where

$$N=\int\exp\Big[-i\int(L_D^{(0)}+L_D^{(1)}+L_G^{(0)})\,d^4x\Big]\delta\psi\delta\bar{\psi}\delta\chi_{\mu\nu}, \tag{8}$$

$$L_D^{(0)}=-\frac{i}{2}\bar{\psi}(\overset{\circ}{\gamma}{}^\alpha\partial_\alpha-\overleftarrow{\partial}_\alpha\overset{\circ}{\gamma}{}^\alpha-2\mu)\psi, \tag{9}$$

$$L_D^{(1)}=\frac{i}{4}\beta\chi_\sigma^\rho\bar{\psi}O_\rho^\sigma\psi, \tag{10}$$

$$O_\rho^\sigma=\delta_\rho^\sigma(\overset{\circ}{\gamma}{}^\alpha\partial_\alpha-\overleftarrow{\partial}_\alpha\overset{\circ}{\gamma}{}^\alpha-2\mu)+(\delta_\rho^\mu\delta_\alpha^\sigma-\tfrac{1}{2}\delta_\alpha^\mu\delta_\rho^\sigma)(\overset{\circ}{\gamma}{}^\alpha\partial_\mu-\overleftarrow{\partial}_\mu\overset{\circ}{\gamma}{}^\alpha), \tag{11}$$

and

$$L_G^{(0)}=-\tfrac{1}{4}\chi^{\alpha\beta}(\delta_\alpha^\mu\delta_\beta^\nu\overset{\circ}{g}{}^{\delta\tau}-\tfrac{1}{2}\overset{\circ}{g}{}_{\alpha\beta}\overset{\circ}{g}{}^{\mu\nu}\overset{\circ}{g}{}^{\tau\delta})\chi_{\mu\nu,\tau\delta}\,. \tag{12}$$

($\overset{\circ}{\gamma}{}^\alpha$ are Dirac's γ^α-matrices in the Minkowski space.) Here the Lagrangian is taken into account to the first-order in β. Later we expand $\exp\Big[-i\int L_{ie}^{(1)}\,d^4x\Big]$ in eq. (7) with respect to β.

§ 3. Propagator of fermion corrected by the second-order self-energy part due to the gravitational field

In order to obtain the second-order self-energy of fermion due to the gravitational field we expand eq. (7) with respect to β and calculate the following expression:

$$S_2'(1,1')=-\tfrac{1}{2}N_0^{-1}\int\psi(1)\bar{\psi}(1')\Big(\frac{i\beta}{4}\Big)^2\chi_\nu^\mu(x)\bar{\psi}(x)O_\mu^\nu(x)\psi(x)\chi_\alpha^\alpha(x')\bar{\psi}(x')O_\alpha^\lambda(x')\psi(x')$$

$$\times\exp\Big[-i(I_G^{(0)}+I_D^{(0)})\Big]\delta\psi\delta\bar{\psi}\delta\chi_{\rho\sigma}d^4xd^4x', \tag{13}$$

where

$$N_0=\int \exp[-i(I_G^{(0)}+I_D^{(0)})]\delta\psi\delta\overline{\psi}\delta\chi_{\rho\sigma}\,. \tag{14}$$

Now the propagators of free fermion and graviton are

$$\begin{aligned} S(1,\,1') &= \{\int \exp[-iI_D^{(0)}]\delta\psi\delta\overline{\psi}\}^{-1}\int \psi(1)\,\overline{\psi}(1')\exp[-iI_D^{(0)}]\delta\psi\delta\overline{\psi} \\ &= S_F(1-1') \end{aligned} \tag{15}$$

and

$$\begin{aligned} R_{\sigma\beta}^{\rho\alpha}(1,\,1') &= \{\int \exp[-iI_G^{(0)}]\delta\chi_{\mu\nu}\}^{-1}\int \chi_\sigma^\rho(1)\,\chi_\beta^\alpha(1')\exp[-iI_G^{(0)}]\delta\chi_{\mu\nu} \\ &= -\frac{i}{2}G_{\sigma\beta}^{\rho\alpha}(1-1')\,, \end{aligned} \tag{16}$$

respectively, where $G_{\sigma\beta}^{\rho\alpha}(x)$ satisfies the following equation:

$$\tfrac{1}{4}(\delta_\alpha^\varkappa\delta_\lambda^\beta-\tfrac{1}{2}\delta_\alpha^\beta\delta_\lambda^\varkappa)\,(-\Box-i\varepsilon)\,G_{\beta\pi}^{\alpha\varphi}(x)=\delta_\pi^\varkappa\delta_\lambda^\varphi\delta(x) \tag{17}$$

and its Fourier transform is

$$G_{\delta\pi}^{\alpha\varphi}(k)=\frac{1}{\pi^2}(\delta_\pi^\alpha\delta_\beta^\varphi-\tfrac{1}{2}\delta_\beta^\alpha\delta_\pi^\varphi)\lim_{\varepsilon\to 0}\frac{1}{k^2-i\varepsilon}\,. \tag{18}$$

Consequently we get

$$S_2'(1,\,1')=\frac{K}{16}\int R_{\sigma\beta}^{\rho\alpha}(x,\,x')\,S(1,\,x)\,O_\rho^\sigma(x)\,S(x,\,x')\,O_\alpha^\beta(x')\,S(x',\,1')\,d^4x d^4x'. \tag{19}$$

Hereof the gravitational self-energy of fermion in a plane wave state $u\exp(-ipx)$ is given by[8]

$$\Delta E=\frac{K}{32}\int G_{\sigma\beta}^{\rho\alpha}(x-x')\,[\,(\bar{u}\exp(ipx)\,O_\rho^\sigma(x)\,S_F(x-x')\,O_\alpha^\beta(x')\,u\exp(-ipx')\,]d^4x \tag{20}$$

$$\equiv(\bar{u}M_s u)\,, \tag{21}$$

where

$$\begin{aligned} M_s=\frac{K}{8\pi}\int &(\delta_\beta^\rho\delta_\sigma^\alpha-\tfrac{1}{2}\delta_\sigma^\rho\delta_\beta^\alpha)\frac{1}{k^2-i\varepsilon}[-\delta_\rho^\sigma\{i\mathring{\gamma}^{\varphi}(2p_\varphi-k_\varphi)+2\mu\} \\ &-i(\delta_\rho^\mu\delta_\varphi^\sigma-\tfrac{1}{2}\delta_\varphi^\mu\delta_\rho^\sigma)\mathring{\gamma}^{\varphi}(2p_\mu-k_\mu)]\frac{i(p_\pi-k_\pi)\mathring{\gamma}^{\pi}-\mu}{(p-k)^2+\mu^2}[-\delta_\alpha^\tau\{i\mathring{\gamma}^{\varrho}(2p_\varrho-k_\varrho)+2\mu\} \\ &-i(\delta_\alpha^\nu\delta_\lambda^\beta-\tfrac{1}{2}\delta_\lambda^\nu\delta_\alpha^\beta)\mathring{\gamma}^{\lambda}(2p_\nu-k_\nu)]d^4k \end{aligned} \tag{22}$$

is calculated using Feynman's cut-off method[8]. Eq. (22) consists of the terms

$$I_n^{(0)}=\int_{-\infty}^{\infty}k_\sigma k_\varrho\ldots k_\mu(k^2-i\varepsilon)^{-1}\{(p-k)^2+\mu^2\}^{-1}d^4k,$$

where n is the power of k in the numerator of the integrand varying from 0 to 3. For instance,

$$I_3(\lambda^2)=\int_{-\infty}^{\infty}k_\sigma k_\tau k_\mu (k^2+\lambda^2-i\varepsilon)^{-1}\{(p-k)^2+\mu^2\}^{-1}d^4k$$

$$=\int_0^1 dx\int_{-\infty}^{\infty}k_\sigma k_\tau k_\mu (k^2-2pkx-\Delta)^{-3}d^4k, \tag{23}$$

where

$$\Delta=-\lambda^2(1-x)-(p^2+\mu^2+i\varepsilon)x+i\varepsilon. \tag{24}$$

If we differentiate eq. (23) two times with respect to λ^2, we get

$$I_3''(\lambda^2)=6\int_0^1 dx\int_{-\infty}^{\infty}k_\sigma k_\tau k_\mu (1-x)^2(k^2-2pkx-\Delta)^{-4}d^4k. \tag{25}$$

After integration over k we have

$$I_3''(\lambda^2)=6\int_0^1 dx(1-x)^2\frac{1}{24i}\left\{-p_\sigma p_\tau p_\mu x^3(p^2x^2+\Delta)^{-2}+\frac{x}{2}A(p^2x^2+\Delta)^{-1}\right\} \tag{26}$$

and

$$I_3(\lambda^2)=\frac{1}{4i}\int_0^1 dx\left[p_\sigma p_\tau p_\mu x^3(\ln D+C\lambda^2+C')+\frac{x}{2}A(D(\ln D-1)+C''\lambda^2+C'''\right], \tag{27}$$

where C, C', C'', C''' are constants and

$$A=p_\sigma\delta_{\tau\mu}+p_\tau\delta_{\sigma\mu}+p_\mu\delta_{\sigma\tau}, \tag{28}$$

$$D=p^2x^2+\Delta=p^2x^2-\lambda^2(1-x)-(p^2+\mu^2+i\varepsilon)x+i\varepsilon. \tag{29}$$

Retaining only $-\lambda^2(1-x)$ in D, we get for $\lambda\gg\mu$, $\lambda\gg p^2$

$$\begin{aligned}I_3(0)-I_3(\lambda^2)=&\frac{1}{4i}p_\sigma p_\tau p_\mu\left[\frac{1}{4}\left(\ln\frac{\lambda^2}{p^2}-C\lambda^2\right)-\frac{19}{48}-\frac{1}{4}\ln\frac{\mu^2}{p^2}+\frac{1}{12}\left(1+\frac{\mu^2}{p^2}\right)\right.\\&\left.+\frac{1}{8}\left(1+\frac{\mu^2}{p^2}\right)^2+\frac{1}{4}\left(1+\frac{\mu^2}{p^2}\right)^3+\frac{1}{4}\left(1+\frac{\mu^2}{p^2}\right)^4\ln\frac{\mu^2}{p^2+\mu^2}\right]\\&+\frac{1}{8i}A\left[\frac{\lambda^2}{6}\ln\lambda^2-\left(\frac{C''}{2}+\frac{5}{36}-\frac{i\pi}{12}\right)\lambda^2-\frac{1}{12}(p^2+4\mu^2)\ln\mu^2\right.\\&-\left\{\frac{1}{4}p^2\left(1+\frac{\mu^2}{p^2}\right)^4-\frac{1}{3}p^2\left(1+\frac{\mu^2}{p^2}\right)^3\right\}\ln\frac{\mu^2}{p^2+\mu^2}-\frac{3}{8}p^2\\&\left.+\frac{17}{36}p^2\left(1+\frac{\mu^2}{p^2}\right)+\frac{1}{24}p^2\left(1+\frac{\mu^2}{p^2}\right)^2+\frac{1}{12}p^2\left(1+\frac{\mu^2}{p^2}\right)^3\right].\end{aligned} \tag{30}$$

Eq. (30) is of the order of p^3 for large momenta p.

In the same way we can calculate $I_2(0)-I_2(\lambda^2)$, $I_1(0)-I_1(\lambda^2)$ and $I_0(0)-I_0(\lambda^2)$ but these terms are of lower order than p^3.

With these terms we calculate M_3, eq. (22), and if there are the terms depending on λ after subtraction corresponding to mass-renormalization, we must impose conditions upon the weight factor $G(\lambda)$ according to Feynman in order to drop the λ-dependent

terms. From eq. (30) $G(\lambda)$ must satisfy the following conditions in the worst case:

$$\int_0^\infty G(\lambda)\,d\lambda=1,\quad \int_0^\infty \lambda^2 G(\lambda)\,d\lambda=\int_0^\infty \lambda^2 \ln\lambda\cdot G(\lambda)\,d\lambda=\int_0^\infty \ln\lambda\cdot G(\lambda)\,d\lambda=0. \tag{31}$$

At any rate we get in this way the finite part of M_s, that is S_c, which behaves like p^3 for large momenta p.

Now if we think any internal fermion line in a given Feynman diagram the total radiative effect from the second-order self-energy parts due to the gravitational field can be taken into account by substituting the following S'_F-function for S_F-function:

$$S'_F(p)=S_F(p)+S_F(p)S_c(p)S_F(p)+\cdots=S_F(p)\{1-S_c(p)S_F(p)\}^{-1}, \tag{32}$$

which is of the order of p^{-3} for large momenta p.

§ 4. Propagators of graviton and boson corrected by the second-order self-energy parts due to fermion and graviton fields respectively

First we calculate the second-order self-energy of graviton. We define the propagator of graviton interacting with the fermion field as follows:

$$\langle T(\chi^\rho_\sigma(1)\chi^\alpha_\beta(1'))\rangle_0=N^{-1}\int \chi^\rho_\sigma(1)\chi^\alpha_\beta(1')\exp[-iI(\psi,\bar\psi,\chi_{\mu\nu})]\,\delta\psi\delta\bar\psi\delta\chi_{\mu\nu} \tag{33}$$

$$=N^{-1}\int \chi^\rho_\sigma(1)\chi^\alpha_\beta(1')\exp[-i\int (L_D^{(0)}+L_D^{(1)}+L_G^{(0)})\,d^4x]\,\delta\psi\delta\bar\psi\delta\chi_{\mu\nu}, \tag{34}$$

where I, N, $L_D^{(0)}$, $L_D^{(1)}$ and $L_G^{(0)}$ are the same as those in Sec. 2.

From eq. (33) we consider the following second-order term

$$S'^{\rho\alpha}_{2\sigma\beta}(1,1')=-\tfrac{1}{2}N_0^{-1}\int \chi^\rho_\sigma(1)\chi^\alpha_\beta(1')\left(\frac{i\beta}{4}\right)^2\chi^{\rho'}_{\sigma'}(x)\bar\psi(x)O^{\sigma'}_{\rho'}(x)\psi(x)$$

$$\times\chi^{\alpha'}_{\beta'}(x')\bar\psi(x')O^{\beta'}_{\alpha'}(x')\psi(x')\exp[-i(I_G^{(0)}+I_D^{(0)})]\,\delta\psi\delta\bar\psi\delta\chi_{\mu\nu}d^4xd^4x'$$

$$=\frac{K}{32}\int R^{\rho\rho'}_{\sigma\sigma'}(1,x)R^{\alpha'\alpha}_{\beta'\beta}(x',1')\bar\psi(x)O^{\sigma'}_{\rho'}(x)S(x,x')O^{\beta'}_{\alpha'}(x')\psi(x')$$

$$\times\exp[-iI_D^{(0)}]\,\delta\psi\delta\bar\psi d^4xd^4x'\{\int\exp[-iI_D^{(0)}]\delta\psi\delta\bar\psi\}^{-1}.$$

With the relations

$$\psi(x)=\int\psi(p)e^{-ipx}d^4p,\quad S(x,x')=\int\frac{i\mathring\gamma^\alpha q_\alpha-\mu}{q^2+\mu^2}e^{-iq(x-x')}d^4q,$$

the above expression becomes

$$\frac{K}{32}\int R^{\rho\rho'}_{\sigma\sigma'}(1,x)R^{\alpha'\alpha}_{\beta'\beta}(x',1')\bar\psi(p)O^{\sigma'}_{\rho'}(\overleftarrow{\partial}_\mu=ip_\mu,\ \partial_\nu=-iq_\nu)\frac{i\mathring\gamma^\nu q_\nu-\mu}{q^2+\mu^2}$$

$$\times O^{\beta'}_{\alpha'}(\overleftarrow{\partial}'_\mu=iq_\mu,\ \partial'_\nu=-ik_\nu)\psi(k)\exp[i(p-q)x+i(q-k)x']\exp[-iI_D^{(0)}]$$

$$\times d^4p d^4q d^4k\,\delta\psi\delta\bar{\psi}d^4xd^4x'\{\int \exp[-iI_D^{(0)}]\delta\psi\delta\bar{\psi}\}^{-1}.$$

We calculate the following expression $S'^{\rho\alpha}_{2\sigma\beta}(1, 1')$ obtained from the above expression by moving $\bar{\psi}(p)$ to the right,

$$S'^{\rho\alpha}_{2\sigma\beta}(1, 1') = -\frac{K}{32}\int R^{\rho\rho'}_{\sigma\sigma'}(1, x)\, R^{\alpha'\alpha}_{\beta'\beta}(x', 1')\, Sp\Big[O^{\alpha'}_{\rho'}(\overleftarrow{\partial}_\mu = ik_\mu,\ \partial_\nu = -iq_\nu)$$

$$\times \frac{i\overset{\circ}{\gamma}{}^{\varphi} q_\varphi - \mu}{q^2+\mu^2} O^{\beta'}_{\alpha'}(\overleftarrow{\partial}'_\mu = iq_\mu,\ \partial'_\nu = -ik_\nu)\frac{i\overset{\circ}{\gamma}{}^{\tau} k_\tau - \mu}{k^2+\mu^2}\Big] \exp[i(k-q)(x-x')]\, d^4qd^4kd^4xd^4x'. \tag{35}$$

When the graviton is in a state χ^ρ_σ with energy-momentum p_μ the diagonal element is given with the relation

$$\int \chi^\alpha_\beta(2)\, R^{\beta\rho'}_{\alpha\sigma'}(2, x)\, d^3x_2 = \chi^{\rho'}_{\sigma'}(x) = \chi^{\rho'}_{\sigma'}(p)\, e^{ipx} + {}^*\chi^{\rho'}_{\sigma'}(p)\, e^{-ipx}$$

as follows:

$$\int \chi^\alpha_\beta(2)\, S'^{\beta\rho}_{2\alpha\sigma}(2, 1)\, \chi^\sigma_\rho(1)\, d^3x_1 d^3x_2$$

$$= -\frac{K}{32}\int \chi^{\rho'}_{\sigma'}(x)\, \chi^{\alpha'}_{\beta'}(x')\, Sp\Big[O^{\sigma'}_{\rho'}(\overleftarrow{\partial}_\mu = ik_\mu,\ \partial_\nu = -iq_\nu)\frac{i\overset{\circ}{\gamma}{}^{\varphi} q_\varphi - \mu}{q^2+\mu^2}$$

$$\times O^{\beta'}_{\alpha'}(\overleftarrow{\partial}'_\mu = iq_\mu,\ \partial_\nu' = -ik_\nu)\frac{i\overset{\circ}{\gamma}{}^{\tau} k_\tau - \mu}{k^2+\mu^2}\Big] e^{i(k-q)(x-x')} d^4qd^4kd^4xd^4x'$$

$$= -\frac{K\pi^4}{2}\int \Big\{ \chi^{\rho'}_{\sigma'}(p)\, \chi^{*\alpha'}_{\beta'}(p)\, Sp\Big[O^{\sigma'}_{\rho'}(\overleftarrow{\partial}_\mu = ik_\mu,\ \sigma_\nu = -i(k_\nu + p_\nu))\frac{i\overset{\circ}{\gamma}{}^{\varphi}(k_\varphi + p_\varphi) - \mu}{(k+p)^2+\mu^2}$$

$$\times O^{\beta'}_{\alpha'}(\overleftarrow{\partial} = i(k_\mu + p_\mu),\ \partial'_\nu = -ik_\nu)\frac{i\overset{\circ}{\gamma}{}^{\tau} k_\tau - \mu}{k^2+\mu^2}\Big] + \chi^{*\rho'}_{\sigma'}(p)\, \chi^{\alpha'}_{\beta'}(p)$$

$$\times Sp\Big[O^{\sigma'}_{\rho'}(\overleftarrow{\partial}_\mu = ik_\mu,\ \partial_\nu = -i(k_\nu - p_\nu))\frac{i\overset{\circ}{\gamma}{}^{\varphi}(k_\varphi - p_\varphi) - \mu}{(k-p)^2+\mu^2}$$

$$\times O^{\beta'}_{\alpha'}(\overleftarrow{\partial}'_\mu = i(k_\mu - p_\mu),\ \partial'_\nu = -ik_\nu)\frac{i\overset{\circ}{\gamma}{}^{\tau} k_\tau - \mu}{k^2+\mu^2}\Big]\Big\} d^4k\int_V d^4x.$$

If we normalize $\chi^\alpha_\beta(x)$ in the volume V, the second-order self-energy of graviton is

$$\begin{aligned} \Delta E &= \chi^{\rho'}_{\sigma'}(p)\, M^{\sigma'\beta'}_{s\rho'\alpha'}(p)\, \chi^{*\alpha'}_{\beta'}(p) + \chi^{*\rho'}_{\sigma'}(p)\, M^{\sigma'\beta'}_{s\rho'\alpha'}(-p)\, \chi^{\alpha'}_{\beta'}(p) \\ &= \{\chi^{\rho'}_{\sigma'}(p)\, \chi^{*\alpha'}_{\beta'}(p) + \chi^{*\rho'}_{\sigma'}(p)\, \chi^{\alpha'}_{\beta'}(p)\}\, M^{\sigma'\beta'}_{s\rho'\alpha'}(p), \end{aligned} \tag{36}$$

where

$$M^{\sigma'\beta'}_{s\rho'\alpha'}(p) = -\frac{iK\pi^4}{2}\int Sp\Big[O^{\sigma'}_{\rho'}(\overleftarrow{\partial}_\mu = ik_\mu,\ \partial_\nu = -i(k_\nu - p_\nu))\frac{i\overset{\circ}{\gamma}{}^{\varphi}(k_\varphi - p_\varphi) - \mu}{(k-p)^2+\mu^2}$$

$$\times O^{\beta'}_{\alpha'}(\overleftarrow{\partial}'_\mu = i(k_\mu - p_\mu),\ \partial'_\nu = -ik_\nu)\frac{i\overset{\circ}{\gamma}{}^{\tau} k_\tau - \mu}{k^2+\mu^2}\Big] d^4k$$

$$= -\frac{iK\pi^4}{2}\int Sp\Big[\{-\delta^{\sigma'}_{\rho'}(-i\mathring{\gamma}^{\alpha}(k_\alpha - p_\alpha) - i\mathring{\gamma}^{\alpha}k_\alpha - 2\mu) + (\delta^{\mu}_{\rho'}\delta^{\sigma'}_{\alpha} - \tfrac{1}{2}\delta^{\mu}_{\alpha}\delta^{\sigma'}_{\rho'})$$

$$\times(-i\mathring{\gamma}^{\alpha}(k_\mu - p_\mu) - i\mathring{\gamma}^{\alpha}k_\mu)\}\frac{i\mathring{\gamma}^{\nu}(k_\nu - p_\nu) - \mu}{(k-p)^2 + \mu^2}\{-\delta^{\beta'}_{\alpha'}(-i\mathring{\gamma}^{\alpha}k_\alpha - i\mathring{\gamma}^{\alpha}(k_\alpha - p_\alpha) - 2\mu)$$

$$+ (\delta^{\mu}_{\alpha'}\delta^{\beta'}_{\nu} - \tfrac{1}{5}\delta^{\mu}_{\nu}\delta^{\beta'}_{\alpha'})(-i\mathring{\gamma}^{\nu}k_\mu - i\mathring{\gamma}^{\nu}(k_\mu - p_\mu))\}\frac{i\mathring{\gamma}^{\tau}k_\tau - \mu}{k^2+\mu^2}\Big]d^4k$$

$$\equiv -16iK\pi^4\int\frac{d^4k}{\left\{\left(k-\frac{p}{2}\right)^2+\mu^2\right\}\left\{\left(k+\frac{p}{2}\right)^2+\mu^2\right\}}[Pk_\alpha k_\beta k_\tau k_\delta + (Q+Q'\mu^2)k_\alpha k_\beta + R\mu^2 + R'\mu^4], \tag{37}$$

where

$$P = \tfrac{3}{4}\delta^{\beta}_{\alpha}\delta^{\delta}_{\tau}\delta^{\sigma'}_{\rho'}\delta^{\beta'}_{\alpha'} + \delta^{\beta}_{\alpha}(\delta^{\beta'}_{\sigma'}\delta^{\tau}_{\rho'}\delta^{\delta}_{\alpha'} - \tfrac{3}{2}\delta^{\sigma'}_{\rho'}\delta^{\tau}_{\alpha'}\delta^{\delta}_{\beta'} - \tfrac{3}{2}\delta^{\beta'}_{\alpha'}\delta^{\tau}_{\rho'}\delta^{\delta}_{\sigma'}) + 2\delta^{\alpha}_{\rho'}\delta^{\beta}_{\sigma'}\delta^{\tau}_{\alpha'}\delta^{\delta}_{\beta'},$$

$$Q = p_\tau p_\delta\{\tfrac{9}{8}(\tfrac{1}{2}\delta^{\tau}_{\delta}\delta^{\beta}_{\alpha} - \delta^{\tau}_{\alpha}\delta^{\delta}_{\beta})\delta^{\sigma'}_{\rho'}\delta^{\beta'}_{\alpha'} + \tfrac{3}{4}\delta^{\tau}_{\alpha}(\delta^{\beta'}_{\alpha'}\delta^{\delta}_{\sigma'}\delta^{\beta}_{\rho'} + \delta^{\sigma'}_{\rho'}\delta^{\delta}_{\beta'}\delta^{\beta}_{\alpha'})$$
$$-\tfrac{1}{3}\delta^{\tau}_{\delta}(\delta^{\beta'}_{\alpha'}\delta^{\alpha}_{\rho'}\delta^{\beta}_{\sigma'} + \delta^{\sigma'}_{\rho'}\delta^{\alpha}_{\alpha'}\delta^{\beta}_{\beta'} - \tfrac{2}{3}\delta^{\beta'}_{\sigma'}\delta^{\alpha}_{\rho'}\delta^{\beta}_{\alpha'}) - \tfrac{1}{2}\delta^{\tau}_{\sigma'}\delta^{\delta}_{\beta'}\delta^{\alpha}_{\rho'}\delta^{\beta}_{\alpha'}\},$$

$$Q' = -\tfrac{1}{2}\delta^{\sigma'}_{\rho'}\delta^{\alpha}_{\alpha'}\delta^{\beta}_{\beta'} - \tfrac{1}{2}\delta^{\beta'}_{\alpha'}\delta^{\alpha}_{\rho'}\delta^{\rho}_{\sigma'} + \delta^{\beta'}_{\sigma'}\delta^{\alpha}_{\rho'}\delta^{\beta}_{\alpha'} + \tfrac{11}{4}\delta^{\beta}_{\alpha}\delta^{\sigma'}_{\rho'}\delta^{\beta'}_{\alpha'},$$

$$R = \tfrac{1}{4}\delta^{\sigma'}_{\rho'}\delta^{\beta'}_{\alpha'}p_\alpha p_\beta,$$

and

$$R' = \delta^{\sigma'}_{\sigma'}\delta^{\beta'}_{\alpha'}.$$

We calculate eq. (37) as before, using Feynman's cut-off method. The result is for $\lambda^2 \gg \mu^2$, $p^2 = 4\mu^2\sin^2\theta$,

$$M^{\sigma'\beta'}_{\rho'\alpha'}(p) = -16iK\pi^4\Big[\frac{iP}{24}\Big[\Big\{\frac{3}{40}(\tfrac{1}{8}Ap^4 - \tfrac{1}{2}Bp^2 - p_\alpha p_\beta p_\tau p_\delta) + \tfrac{1}{4}(\tfrac{1}{2}Ap^2 + B)(\mu^2 + \tfrac{1}{4}p^2)$$
$$-\tfrac{3}{4}A(\mu^2+\tfrac{1}{4}p^2)^2\Big\}\ln\frac{\mu^2}{\lambda^2} - \tfrac{1}{2}\{\tfrac{1}{4}Ap^2 + \tfrac{1}{2}B - 3A(\mu^2+\tfrac{1}{4}p^2)\}\lambda^2\ln\lambda^2 + \tfrac{3}{4}A\lambda^4\ln\lambda^2$$
$$+\Big\{-(\mu^2+\tfrac{1}{4}p^2)^3\theta\cdot\tan\theta + \frac{p^2}{20}(\mu^2+\tfrac{1}{4}p^2)(\mu^2+\tfrac{1}{2}p^2)\Big\}$$
$$\times\Big\{\frac{6}{5}(Ap^4 - 4Bp^2 - 8p_\alpha p_\beta p_\tau p_\delta)\frac{1}{p^6} + 4(Ap^2+2B)\frac{1}{p^4} - 6A\frac{1}{p^2}\Big\} - C\lambda^4 - C'\lambda^2\Big]$$
$$-\frac{i}{8}Q\Big[\delta_{\alpha\beta}\Big\{(\mu^2+\lambda^2)\ln\Big(1+\frac{\lambda^2}{\mu^2}\Big) + C''\lambda^2\Big\} - (p_\alpha p_\beta - \delta_{\alpha\beta}p^2)$$
$$\times\Big\{\tfrac{1}{9} + \frac{4\mu^2+p^2}{p^2}(\tfrac{1}{3} - \theta\cot\theta) + \tfrac{1}{6}\ln\frac{\lambda^2}{\mu^2}\Big\}\Big]$$
$$+Q'\Big[\mu^2\Big\{-\frac{i}{8}\delta_{\alpha\beta}\Big((\mu^2+\lambda^2)\ln\Big(1+\frac{\lambda^2}{\mu^2}\Big) + C'''\lambda^2\Big) + \frac{i}{8}(p_\alpha p_\beta - \delta_{\alpha\beta}p^2)$$
$$\times\Big(\tfrac{1}{6}\ln\frac{\lambda^2}{\mu^2} + \tfrac{1}{9} + \frac{4\mu^2+p^2}{3p^2}(\tfrac{1}{3} - \theta\cot\theta)\Big)\Big\}$$

$$+\frac{i}{4}\lambda^2\Big\{-(\tfrac{1}{2}p_\alpha p_\beta+\tfrac{1}{4}\delta_{\alpha\beta}p^2)\Big(\tfrac{1}{9}+\frac{p^2+4\mu^2+4\lambda^2}{3p^2}\Big)+\delta_{\alpha\beta}(\tfrac{1}{3}p^2+\tfrac{3}{2}\mu^2+\tfrac{3}{2}\lambda^2)+C_0\lambda^2+C_0'$$

$$+\ln(-\mu^2-\lambda^2)\Big(\frac{1}{12}p_\alpha p_\beta-\tfrac{1}{2}\delta_{\alpha\beta}(\tfrac{1}{6}p^2+\mu^2+\lambda^2)\Big)+\tfrac{2}{9}p^2\Big(\mu^2+\frac{p^2}{2}\Big)\Big(-\frac{p_\alpha p_\beta}{p^4}+\delta_{\alpha\beta}\frac{1}{p^2}\Big)\Big\}\Big]$$

$$+\frac{i}{8}R\Big[4\mu^2(1-\theta\cot\theta)-\tfrac{1}{3}p^2+2(\lambda^2+\mu^2)\ln\Big(1+\frac{\lambda^2}{\mu^2}\Big)-C_0''\lambda^2\Big]$$

$$+\frac{i}{4}R'\Big[\mu^4\Big\{\ln\frac{\lambda^2}{\mu^2}+2(1-\theta\cot\theta)\Big\}+(2\mu^2\lambda^2+\lambda^4)\{\ln(-\mu^2-\lambda^2)+C_0'''\}\Big]\Big], \tag{38}$$

where

$$A=\delta_{\alpha\delta}\delta_{\beta\tau}+\delta_{\beta\delta}\delta_{\alpha\tau}+\delta_{\tau\delta}\delta_{\alpha\beta},$$

$$B=\delta_{\alpha\delta}p_\beta p_\tau+\delta_{\beta\delta}p_\alpha p_\tau+\delta_{\tau\delta}p_\alpha p_\beta+\delta_{\beta\tau}p_\alpha p_\delta+\delta_{\alpha\tau}p_\beta p_\delta+\delta_{\alpha\beta}p_\tau p_\delta$$

and C, C', C'', C''', C_0, C_0', C_0'' and C_0''' are functions of p independent of λ.

In eq. (38) if there are the terms depending on λ^2 after subtraction of mass- and coupling-renormalization, we must impose conditions upon the weight factor $G(\lambda)$ in order to drop the λ-dependent terms. From eq. (38) $G(\lambda)$ must satisfy the following conditions in the worst case:

$$\int_0^\infty G(\lambda)\,d\lambda=1,\quad \int_0^\infty\lambda^2G(\lambda)\,d\lambda=\int_0^\infty\lambda^4G(\lambda)\,d\lambda=\int_0^\infty G(\lambda)\ln\lambda d\lambda$$

$$=\int_0^\infty\lambda^2G(\lambda)\ln\lambda\,d\lambda=\int_0^\infty\lambda^4G(\lambda)\ln\lambda\,d\lambda=0. \tag{39}$$

In this way we obtain the finite part of $M_{\rho\alpha\sigma'\beta'}^{\alpha'\beta'}$, that is $R_{\rho\alpha\sigma'\beta'}^{\alpha'\beta'}$, which behaves like p^4 for large momenta p. As in Sec. 3 the effect of second-order self-energy part is given by substituting the following $R'^{\alpha\beta}_{\rho\alpha}$-function for $R^{\alpha\beta}_{\rho\alpha}$-function:

$$R'^{\alpha\beta}_{\rho\alpha}(p)=R^{\alpha\beta}_{\rho\alpha}(p)+R^{\alpha\beta'}_{\rho\alpha'}(p)\,R^{\alpha'\beta''}_{\alpha\beta''\alpha'}(p)\,R^{\alpha''\beta}_{\alpha\beta''\alpha}(p)+\cdots=R^{\alpha\nu}_{\mu\mu}(p)\{1-R_c(p)\,R(p)\}^{-1}{}^{\mu\rho}_{\nu\alpha}, \tag{40}$$

which is of the order of p^{-4} for large momenta p.

Next in order to obtain the propagator of boson corrected by the second-order self-energy part due to the gravitational field we define the propagator as follows:

$$\langle T(\phi(1)\phi(1'))\rangle_0=N^{-1}\int\phi(1)\phi(1')\exp[-iI(\phi,\boldsymbol{g}_{\mu\nu})]\partial\phi\,\partial\boldsymbol{g}_{\mu\nu}, \tag{41}$$

where

$$L=\sqrt{g}\,\{\beta^{-2}R+\tfrac{1}{2}(-\phi\Box\phi+\kappa^2\phi^2)\}. \tag{42}$$

If we write the second-order self-energy of boson due to the gravitational field as

$$\Delta E=\phi^*(p)\,M_s(p)\,\phi(p), \tag{43}$$

$M_s(p)$ can be calculated as before. In this case the second-order derivative coupling appears in the power series expansion and the finite part of $M_s(p)$ obtained after sub-

traction corresponding to the self-mass renormalization, that is $\Delta_c(p)$, behaves like p^4 for large momenta p, and the corrected propagator

$$\Delta'_F(p)=\Delta_F(p)\{1-\Delta_c(p)\Delta_F(p)\}^{-1} \tag{44}$$

is of the order of p^{-4} for large momenta p.

§ 5. S-matrix

When we calculate an S-matrix we must of course use the Lagrangian including all powers of $\beta\chi_{\mu\nu}$. But since each term of this Lagrangian is at most of the second order with respect to powers of differential operator we can investigate the convergence problem of all Feynman diagrams by making use of the following quantities :

n : the number of vertices in a given Feynman diagram,
n_1 : the number of vertices where the first-order derivative coupling appears between graviton and fermion for their internal lines,
n_2 : the number of vertices where the second-order derivative coupling appears between graviton and boson for their internal lines,
n_3 : the number of vertices where the second-order derivative coupling appears between graviton and graviton for their internal lines,
F_f : the number of internal fermion lines,
F_g : the number of internal graviton lines,
F_b : the number of internal boson lines,
F : the number of energy-momentum integrals for internal lines.
(The external lines are not taken into account in the following dicussion.)

If we calculate an S-matrix with the above corrected propagators and if there is no derivative coupling between fermion and boson, the condition of convergence for the given Feynman diagram is

$$-1\geq -3F_f-4F_g-4F_b+n_1+2n_2+2n_3+4F.$$

As F_f is less than or equal to n we can write

$$F_f=n-a \quad (a\geq 0)$$

and F is given by the relation

$$F=F_f+F_b+F_g-n+1,$$

so that the above condition becomes as follows :

$$3n+a-n_1-2n_2-2n_3\geq 5 \quad (a\geq 0). \tag{45}$$

Eq. (45) is satisfied except for the following cases :

The cases of divergence : $n\leq 4$,

(i) $n=4,\ n_2+n_3=4,\ a=0\ \ (F_f=4)$,

(ii) $n=3,\ n_2+n_3=3,\ a=1\ \ (a=0$ cannot occur.),

(iii) $n=2,\ n_2+n_3=2$.

(i) In this case the four internal fermion lines form the two closed loops and at all four vertices the second-order derivative couplings appear. But from the total Lagrangian density

$$L=L_G+L_{GD}+L_{GB}+L_{GBD},$$

where

$$L_G=\sqrt{g}\,\beta^{-2}R,\quad L_{GD}=-i\sqrt{g}\,\bar{\psi}(\gamma^a\nabla_a-\mu)\psi,$$

$$L_{GB}=-\tfrac{1}{2}\sqrt{g}\,(\phi\sqrt{g}\,\partial_\mu(\sqrt{g}\,g^{\mu\nu}\partial_\nu)\phi-\kappa^2\phi^2),\quad L_{GBD}=ig_0\sqrt{g}\,\bar{\psi}\phi\psi$$

(g_0 is a coupling constant between fermion and boson. L_{GBD} takes the other form according to the kind of boson.), there are no terms which contribute to this case, because those terms must be the products of fermion field and the second-order derivatives of graviton field or boson field while the latter appear only in L_G and L_{GB} which have no fermion field.

(ii) This case does not occur, too, because of the above reason.

(iii) (a) The case $a=0$ or 1 does not occur, too, because of the above reason. (b) The case $a=2$ corresponds to the graviton self-energy due to boson or to the graviton-graviton scattering containing boson closed loop or graviton internal lines.

Consequently we have, in the S-matrix, the divergent cases which have the number of vertices $n=2$ and correspond to (I) the graviton self-energy due to boson (Fig. 1 (1) is the simplest case. This may, however, be dropped because of the gauge invariance.) and to (II) the graviton-graviton scattering containing boson closed loop (Fig. 1 (2) is the simplest case.) or graviton internal lines (Fig. 1 (3) are the simplest cases.).

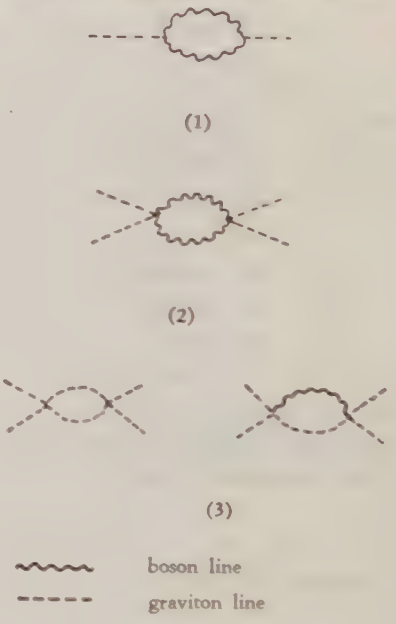

Fig. 1.

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Effect of the Non-relativistic Recoil of a Source Particle in Quantum Field Theories

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The purpose of this paper is to develop a calculation method convenient for the treatment of polaron-like systems, that is, the systems of a quantized field and a non-relativistically moving particle which are interacting fairly strongly with each other. Firstly, generalizing the Chew-Low method, we construct a set of equations by which we can directly calculate the effect of the recoil of source-particle to the scattering amplitudes. A formula for the effective mass of particle is also given. Then, four problems are solved by our method, as examples. The former two are the scattering of neutral scalar meson and the effective mass of polaron. Though these were already solved, we reinvestigate them in order to explain and justify our approximation method. The latter two are the *P*-wave scattering of the charged scalar meson and of the pair-theory meson. Finally, we discuss the importance of the non-relativistic recoil.

§ 1. Introduction

The study of the tight interaction between quantized fields and a non-relativistically moving particle is currently receiving considerable interest. This interest has been spurred mainly by pion-nucleon problems and by polaron problems.

The treatment of the unweakly interacting systems is very difficult when the perturbation method fails. In order to solve these problems, various calculation methods have been presented until now; for examples, Tomonaga's intermediate coupling theory,[1] the Tamm-Dancoff method,[2] the Chew-Low method,[3] etc. Particularly the Chew-Low method is a very excellent one, because the symmetries in the original Hamiltonian are preserved to the last and only the renormalized quantities occur in the final results. But all these methods are powerful only in the static approximation in which the recoil of particle is neglected.

In the polaron problem, the neglect of the recoil of particle is quite meaningless, because it is the kinetic energy of the particle itself to be sought. In the case of the pion physics, there are still some outstanding puzzles, for example, the strong isotopic dependence of the *S*-wave phase shifts in pion-nucleon scattering[4] —these also force us to doubt the applicability of the static approximation. But we have not yet any consistent methods which are effective when neither perturbation method nor static approximation

is applicable. Therefore, we here develop a systematic calculation method convenient for these cases.

In the pion physics, the recoil often means the nucleon pair formation. This kind of recoil can be partly taken into account by using the Foldy and Tani transformation[2] which eliminates the negative energy states of the nucleon. In this paper, however, we do not discuss such a pair formation but the simple translation of particle.

In Sec. 2–A, we take up an arbitrary Hamiltonian, provided that it does not contain any velocity-dependent interaction term. It is assumed that the particle in the system may move non-relativistically but no particle-antiparticle pair is created. This Hamiltonian is then transformed into the representation in which the total momentum of the system is diagonal, by using the Jost-transformation.[6] In Sec. 2–B, we apply the Chew-Low method[3] to this transformed Hamiltonian and construct a set of equations for the generalized scattering amplitudes. This generalization is necessary to calculate the effective mass of particle. In Sec. 2–C, the scattering amplitudes are splitted into two parts, the static part and the recoil part, and the equations for the recoil part are sought. It is shown there that these equations are obtained without referring to the details of the interaction, if the static parts of scattering amplitudes are closely known. (The static part is the part which is obtained in the static approximation, and we assume throughout this paper that this part is completely known.) The equations thus obtained can sometimes be solved easily in the one-meson approximation. For some examples, we shall illustrate these circumstances in the later section. In Sec. 2–D, some discussions on the effective mass of the particle are presented.

In Sec. 3–A, we discuss the scattering of the neutral scalar mesons. In the static approximation, the neutral scalar mesons are not scattered at all. But, if we take account of the recoil of particle, the P-wave scattering occurs. The solution of this problem was already given by McVoy and Steinwedel.[7] They used the canonical transformation which was studied by van Kampen[8] in relation with the light scattering. This method is very clever, but the physical meaning of the approximation is somewhat vague. We show, applying the method presented here, that the solution is very easily obtained and the meaning of the approximation is clarified. In Sec. 3–B, we calculate the effective mass of the polaron in the same approximation as in the case of the neutral scalar meson field. This problem was formerly solved by Lee, Low and Pines.[9] They applied the variation method to the Jost-transformed Hamiltonian. The results obtained here agree with their results.

These two examples were discussed to justify the present approximation by comparing our results with the solutions by other methods. Next we discuss two other examples which have not been investigated yet.

In Sec. 3–C, we discuss the scattering of the charged scalar mesons. The mesons of this type are scattered only from the S-wave state in the static approximation. But, if the nucleon recoil is taken into account, the P-wave scatterings also occur. We there study this P-wave scattering, applying the present method which is well justified by the preceding two examples. In this case, though we have no exact solutions even in the

static limit, a good solution in one-meson approximation is available which is known as the Lee-Serber solution.[10] Using these approximate scattering amplitudes, we make the equations for recoil parts, but the equations are too complicated to be exactly solved. However, we can easily obtain the closely approximate solutions, because we may use of the fact that the effect of virtually propagating *P*-wave mesons is negligible compared with that of *S*-wave mesons. The fact is proved there by solving the simplified equations obtained by a modification of the inhomogeneous terms in original equations. In Sec. 3–D, the *P*-wave scatttering of mesons in the pair-theory[11] is discussed briefly. In this case, the circumstance is quite similar to the charged scalar case. The *P*-wave phase shift is calculated there in the same approximation as Sec. 3–C.

Finally in Sec. 4, we summarize the main results of this paper and present some discussions on the importance of non-relativistic recoils, specially of considering them to the higher order of coupling constant. For example, the strange dependences on energy and coupling-constant of the recoil parts of phase shifts are discussed, which cannot be even imagined in the perturbation theory.

§ 2. General theory

A. Hamiltonian

We first take as our Hamiltonian

$$\boldsymbol{H}=\boldsymbol{H}_0+\boldsymbol{H}_I(\boldsymbol{r})+\boldsymbol{H}_p. \tag{1}$$

Here $\boldsymbol{H}_0$ and $\boldsymbol{H}_p$ are the unperturbed Hamiltonians of a field and a particle respectively, that is,

$$\boldsymbol{H}_0=\sum_k \omega_k a_k^{+} a_k$$

and

$$\boldsymbol{H}_p=\boldsymbol{p}^2/2m,$$

where ω_k is the energy of a field quantum with a momentum k, a_k^{+} and a_k are creation and annihilation operators for single quanta, respectively, $\boldsymbol{p}$ is the momentum of particle and m is its unrenormalized mass. $\boldsymbol{H}_I(\boldsymbol{r})$ is the interaction Hamiltonian. We do not put on $\boldsymbol{H}_I$ any limitations except that (i) it contains the position vector of particle $\boldsymbol{r}$ in the forms $a_k e^{ik\cdot r}$ and $a_k^{+} e^{-ik\cdot r}$ but (ii) not the momentum $\boldsymbol{p}$ at all. (The condition (ii) is not essential for the present work but we require this for simplicity.)

It is, however, very difficult to work directly with such a Hamiltonian containing both $\boldsymbol{p}$ and $\boldsymbol{r}$. In order to remove this difficulty, we eliminate $\boldsymbol{r}$ by making use of the Jost-transformation.[6] Transforming (1) by a unitary operator

$$U=\exp\left(i\sum_k \boldsymbol{k}\cdot\boldsymbol{r}\, a_k^{+} a_k\right),$$

we obtain

$$\begin{aligned} H_T&=U\boldsymbol{H}U^{-1}\\ &=H+H'+H_P, \end{aligned} \tag{2}$$

where

$$H=H_0+H_s+H_r, \tag{3}$$

$$H_0=\sum_k E_k a_k^+ a_k, \quad E_k=\omega_k+(k^2/2m),$$

$$H_s=H_I(0)$$

$$H_r=(1/2m)\sum_{kk'}\boldsymbol{k}\cdot\boldsymbol{k}' a_{k'}^+ a_k^+ a_{k'} a_k$$

and

$$H_P=\boldsymbol{P}^2/2m, \quad H'=-\sum_k(1/m)\boldsymbol{P}\cdot\boldsymbol{k}\, a_k^+ a_k. \tag{4}$$

Instead of (1), we use (2)–(4) in the following sections. In (4), $\boldsymbol{P}$ is the resultant momentum of field and particle. It is unnecessary to move the center of mass, except for the calculation of the effective mass of particle. Thus, we use only H of (3) until Sec. 2–D.

B. Generalized Chew-Low equations

We next derive the generalized Chew-Low equations from the Hamiltonian of (3). The formal solutions of the Schrödinger equation,

$$H\Psi_n=E_n\Psi_n, \quad n=(p_1,\cdots p_n), \tag{5}$$

are

$$\Psi_n^{(\pm)}=N_n a_{p_1}^+ a_{p_2}^+\cdots a_{p_n}^+\Psi_0-\frac{1}{H-E_n\mp i\epsilon}U_n\Psi_0, \tag{6}$$

where $(\pm)$ on Ψ_n is concerned with the boundary condition at infinity as usual, Ψ_n denotes the single-particle state, N_n is a normalization constant,

$$U_n=N_n[H_s+H_r,\ a_{p_1}^+ a_{p_2}^+\cdots a_{p_n}^+] \tag{7}$$

and

$$E_n=\sum_{i=1}^{n}E_{p_i}. \tag{8}$$

We note that, strictly speaking, Ψ_n does not denote the n-quantum state, since E_n is not equal to the energy of this state

$$E_n'=\sum_i\omega_{p_i}+(\sum_i p_i)^2/2m^*,$$

where m^* is the effective mass of particle. In the case of sufficiently low energies, however, we may regard it as denoting such a state.

Using these state vectors, we define the generalized scattering amplitudes

$$T_l(n,\ m)=(\Psi_n^{(-)}U_l\Psi_m^{(-)}) \quad \text{and} \quad S_l(n,\ m)=(\Psi_n^{(-)}U_l^+\Psi_m^{(-)}). \tag{9}$$

Particularly,

$$T_l(n,\ 0)=T_l(n) \quad \text{and} \quad S_l(n,\ 0)=S_l(n) \tag{10}$$

are reduced to the scattering amplitudes of Chew and Low in the limit $m \to \infty$.

The equations for $T_l(n, m)$ and $S_l(n, m)$ are easily obtained. Substituting (6) in (9), we have

$$T_l(n, m) = (\Psi_0 w_{nl} \Psi_m^{(-)}) + (\Psi_0 U_l N_n a_{p_1} \cdots a_{p_n} \Psi_m^{(-)}) - \left(\Psi_0 U_n^+ \frac{1}{H - E_n - i\epsilon} U_l \Psi_m^{(-)}\right), \tag{11}$$

where

$$w_{nl} = N_n[a_{p_1} \cdots a_{p_n}, U_l]. \tag{12}$$

Making use of the completeness of $\Psi_s^{(-)}$ and the formula proved in Appendix

$$N_n a_{p_1}^+ \cdots a_{p_n}^+ \Psi_s^{(-)} = \frac{N_n N_s}{N_{n+s}} \Psi_{n+s}^{(-)} + \frac{1}{H - E_{n+s} + i\epsilon} U_n \Psi_s^{(-)}, \tag{13}$$

where $n+s = (p_1, \cdots p_n, q_1 \cdots q_s)$ if $n = (p_1, \cdots p_n)$ and $s = (q_1 \cdots q_s)$, we can write (11) as

$$T_l(n, m) = (\Psi_0 w_{nl} \Psi_m^{(-)}) + \sum_s S_l(s) \frac{N_n N_s}{N_m} \delta_{m, s+n} - \sum_s \left[\frac{T_n(s) T_l(s, m)}{E_s - E_n - i\epsilon} + \frac{S_l(s) S_n(s, m)}{E_s + E_n - E_m + i\epsilon}\right]. \tag{14}$$

Quite similarly,

$$S_l(n, m) = (\Psi_0 \tilde{w}_{nl} \Psi_m^{(-)}) + \sum_s T_l(s) \frac{N_n N_s}{N_m} \delta_{m, s+n} - \sum_s \left[\frac{T_n(s) S_l(s, m)}{E_s - E_n - i\epsilon} + \frac{T_l(s) S_n(s, m)}{E_s + E_n - E_m + i\epsilon}\right], \tag{15}$$

where

$$\tilde{w}_{nl} = N_n[a_{p_1} \cdots a_{p_n}, U_l^+]. \tag{16}$$

Eqs. (14) and (15) are the generalized Chew-Low equations used below.

C. Recoil effects on the scattering amplitudes

We first consider the equations

$$T_m^s(n) = (\Psi_0 w_{nm}^s \Psi_0) - \sum_l \left[\frac{T_n^s(l) T_m^s(l)}{E_l - E_n - i\epsilon} + \frac{S_m^s(l) S_n^s(l)}{E_l + E_n}\right] \tag{17}$$

and

$$S_m^s(n) = (\Psi_0 \tilde{w}_{nm}^s \Psi_0) - \sum_l \left[\frac{T_n^s(l) S_m^s(l)}{E_l - E_m - i\epsilon} + \frac{T_m^s(l) S_n^s(l)}{E_l + E_n}\right], \tag{18}$$

where

$$w_{nm}^s = N_n[a_{p_1} \cdots a_{p_n}, U_m^s], \tag{19}$$

$$U_m^s = N_m[H_s,\ a_{p_1}^+ \cdots a_{q_m}^+] \tag{20}$$

and $\tilde{w}_{nm}^{\ s}$ is obtained from (19) by the replacement of U_m^s by U_m^s. From these equations we can determine all of T^s's and S^s's, once we give as the boundary condition the values of $(\Psi_0 w_{p'l'}^{\ s} \Psi_0)$, $T_{p'}^{\ s}(0)$, etc., which are easily obtained directly from their definitions.

Eqs. (17) and (18) are different from the ordinary Chew-Low equations in two respects: (i) Ψ_0 is the lowest-eigenvalue state of $H_v + H_c + H_r$ (not of $H_v + H_c$) and (ii) E_n is defined by (8) (not equal to $\sum_i \omega_i$ used in the static approximation).

Owing to (i), the values of present $T_{p'}^{\ s}(0)$, etc., and those in the static approximation may be different from each other by their proportional factors. From the first, however, these factors are undetermined, which are determined appropriately as the renormalized coupling constants. Hence the difference (i) is trivial. The difference (ii) is also unimportant, since we may often solve these equations if we can solve the corresponding ordinary Chew-Low equations and moreover the substitution of $\sum_i \omega_i$ for E_n causes no remarkable errors except for some special cases. (c. f. Sec. 3-B). By these reasons we may regard (17) and (18) as the *static* Chew-Low equations of which the solutions are assumed to be completely known.

Next we define the recoil parts, $T_m^r(n)$ and $S_m^r(n)$, by

$$T_m^r(n) = T_m(n) - T_m^s(n) \quad \text{and} \quad S_m^r(n) = S_m(n) - S_m^s(n). \tag{21}$$

(Correspondingly $T_m^s(n)$ and $S_m^s(n)$ are hereafter called the static parts.) The equations for $T_m^r(n)$ and $S_m^r(n)$ are easily obtained from (14), (15), (17), and (18):

$$\begin{aligned} T_m^r(n) = (\Psi_0 w_{nm}^{\ r} \Psi_0) - \sum_l' \Bigg[& \frac{T_n^s(l) + T_m^r(l) + T_n^r(l) + T_m^s(l)}{E_l - E_n - i\epsilon} \\ & + \frac{S_m^s(l) + S_n^r(l) + S_m^r(l) + S_n^s(l)}{E_l + E_n} \Bigg] \\ - \sum_l' \Bigg[& \frac{T_n^r(l) + T_m^r(l)}{E_l - E_n - i\epsilon} + \frac{S_m^r(l) + S_n^r(l)}{E_l + E_n} \Bigg] \end{aligned} \tag{22}$$

and

$$\begin{aligned} S_m^r(n) = (\Psi_0 \tilde{w}_{nm}^{\ r} \Psi_0) - \sum_l' \Bigg[& \frac{T_n^s(l) + S_m^r(l) + T_n^r(l) + S_m^s(l)}{E_l - E_n - i\epsilon} \\ & + \frac{T_m^s(l) + S_n^r(l) + T_m^r(l) + S_n^s(l)}{E_l + E_n} \Bigg] \\ - \sum_l' \Bigg[& \frac{T_n^r(l)\ S_m^r(l)}{E_l - E_n - i\epsilon} + \frac{T_m^r(l)\ S_n^r(l)}{E_l + E_n} \Bigg] \end{aligned} \tag{23}$$

where

$$w_{nm}^r = N_n[a_{p_1} \cdots a_{p_n},\ U_m^r], \tag{24}$$

$$U_m^r = N_m[H_r,\ a_{q_1}^+ \cdots a_{q_m}^+] \tag{25}$$

and $\tilde{w}_{nm}^{\ r}$ is obtained by the replacement of U_m^r by U_m^{r+}. Using (22) and (23), we can

directly calculate the recoil effect to the scattering amplitude.

These equations are, in a sense, only the réchauffé of Chew-Low's, but very convenient from the practical view-point, since these do not depend explicitly on the character of the system or the structure of interaction Hamiltonian. (They enter the equations only through T^s and S^s, which are assumed to be completely known.) For example, we can write out $(\Psi_0 w_{pq}^r \Psi_0)$ only in terms of T and S without referring to the detail of H_I:

$$(\Psi_0 w_{pq}^r \Psi_0) = \left(\Psi_0 | \delta_{pq} \frac{1}{m} \Sigma' \boldsymbol{q}\cdot\boldsymbol{k} a_k^+ a_k + \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} a_q^+ a_p | \Psi_0 \right)$$

$$= \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \left(\Psi_0 U_q \frac{1}{H+E_q} \frac{1}{H+E_p} U_p^+ \Psi_0 \right)$$

$$= \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \sum_l \frac{S_q(l)^* S_p(l)}{(E_l+E_q)(E_l+E_p)}, \tag{26}$$

where we have used (6), (9) and a relation $(\Psi_0 \Sigma'_k \boldsymbol{k} a_k^+ a_k \Psi_0) = 0$. Similarly,

$$(\Psi_0 \tilde{w}_{pq}^r \Psi_0) = \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \left\{ - \frac{S_q(p)}{E_p+E_q} - \sum_l \frac{T_p(l)^* S_q(l)}{(E_l-E_p-i\epsilon)(E_l+E_q)} \right\}. \tag{27}$$

In Sec. 3 we really show that various problems can be easily calculated by using the equations obtained here.

D. Effective mass

Before going to the applications of the equations obtained in the preceding section, we here briefly develop a formula for the effective mass of particle. The effective mass of particle is obtained easily by calculating the energy shift caused by H' of (4). The second-order energy shift is

$$\Delta E = -\sum_{n\neq 0} \frac{|(\Psi_n^{(-)}|H'|\Psi_0)|^2}{E_n}. \tag{28}$$

Here,

$$(\Psi_n^{(-)}|H'|\Psi_0) = \sum_k \frac{1}{m} (\Psi_n^{(-)} \boldsymbol{P}\cdot\boldsymbol{k} a_k^+ a_k \Psi_0)$$

$$= -\sum_{k,s} \frac{1}{m} \boldsymbol{P}\cdot\boldsymbol{k} \frac{1}{E_s+E_k} \left\{ \frac{N_s}{N_n} \delta_{n,s+k} + \frac{T_k(n,s)}{E_n-E_s-E_k+i\epsilon} \right\} S_k(s), \tag{29}$$

where we have made use of eq. (13). Performing the summation over the virtual states, we may reduce (29) to the form $\Delta E = \boldsymbol{P}^2 C$. Combining this with the definition of effective mass m^*, we obtain

$$1/2m^* - 1/2m = C.$$

By using this formula, the effective mass of the nucleon interacting with the neutral scalar meson field and that of the polaron shall be actually calculated in Sec. 3–A and Sec. 3–B, respectively.

§ 3. Applications

In this section, we apply our method to four simple examples. The first two are the scattering of a neutral scalar meson and the effective mass of a polaron. These were formerly solved by McVoy and Steinwedel[7] (using the canonical transformation method) and Lee, Low and Pines[9] (using the variation method), respectively. We here reinvestigate these for the illustration and justification of our method.

The latter two examples are the *P*-wave scatterings in the charged scalar meson theory and Wentzel's meson pair theory.[11] In the static approximation, only the *S*-wave scatterings occur for these cases. We here investigate the *P*-wave scattering due to recoils of particle.

A. Scattering of a neutral scalar meson

We here discuss the scattering of a neutral scalar meson by a nucleon. This problem was before discussed by McVoy and Steinwedel. They first applied the so-called dipole approximation presented by van Kampen to the interaction Hamiltonian, namely, expanded it with respect to $\boldsymbol{r}$ to the second order

$$\boldsymbol{H}_I(\boldsymbol{r}) \simeq \boldsymbol{H}_I(0) + \sum_i \left(\frac{\partial \boldsymbol{H}_I}{\partial x_i} \right)_{r=0} \cdot x_i - \frac{1}{2} \sum_{i,j}' \left(\frac{\partial^2 \boldsymbol{H}_I}{\partial x_i \partial x_j} \right)_{r=0} x_i x_j ,$$

and then diagonalized this approximated Hamiltonian by making use of two successive unitary transformations. However, this approximation method is somewhat vague in its physical meaning. Here we show that the solution for this system is easily obtained from our equations by making use of one-meson approximation, of which the physical meaning has been well established.[12]

The interaction Hamiltonian for the neutral scalar meson theory is

$$\boldsymbol{H}_I(\boldsymbol{r}) = \sum_k V_k (a_k e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + a_k^+ e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}) ,$$

$$V_k = g \frac{\tau(k)}{\sqrt{2\omega_k}} , \tag{31}$$

where $\tau(k)$ is the cut-off factor and $\omega_k = \sqrt{k^2 + \mu^2}$ (μ is the meson mass). The static Chew-Low equations for this interaction can be easily solved, since it follows directly from the definitions that

$$T_q^s(0) = S_q^s(0) = V_q . \tag{32}$$

The solutions are

$$T_q^s(m) = S_q^s(m) = 0, \quad (m \neq 0) . \tag{33}$$

(We do not discuss here the multiplicity of solutions.[13]) Using these results in (26) and (27), we have

$$(\Psi_0 \boldsymbol{w}_{pq}^r \Psi_0) = \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \frac{V_p V_q}{E_p E_q} + \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \sum_{l \neq 0} \frac{S_q^r(l) + S_p^r(l)}{(E_l + E_q)(E_l + E_p)} \tag{34}$$

and

$$(\Psi_0 \tilde{w}_{uq}^{r} \Psi_0) = \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \frac{V_p V_q}{E_p E_q} - \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \frac{S_q^r(p)}{E_p+E_q} - \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \times \sum_{l\neq 0} \frac{T_p^r(l)+S_q^r(l)}{(E_l-E_p-i\epsilon)(E_l+E_q)}, \tag{35}$$

where the easily proved relations $T_p^r(0)=S_p^r(0)=0$ have been used.

In order to calculate the main P-wave amplitude, the second term of (34) and the second and third terms of (35) are unnecessary. Strictly speaking, we must also take these terms, because T^r and S^r contain the S-wave parts due to nucleon-recoil. However, these terms are of higher order in $1/m$ and we may neglect them. (It is not the higher-order terms in $1/m$ but those in g^2/m that we calculate below in (40), etc.) Dropping these terms, we obtain the equations for the P-wave scattering amplitude in the one-meson approximation as follows:

$$T_q^r(p) = \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \frac{V_p V_q}{E_p E_q} - \sum_k \left[\frac{T_p^r(k) - T_q^r(k)}{E_k-E_p-i\epsilon} + \frac{S_q^r(k)+S_p^r(k)}{E_k+E_p} \right] \tag{36}$$

and

$$S_q^r(p) = \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \frac{V_p V_q}{E_p E_q} - \sum_k \left[\frac{T_p^r(k)+S_q^r(k)}{E_k-E_p-i\epsilon} + \frac{T_q^r(k)+S_p^r(k)}{E_k+E_p} \right]. \tag{37}$$

Setting

$$T_q^r(p) = S_q^r(p) = \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m} \frac{V_p V_q}{E_p E_q} h(\omega_p), \tag{38}$$

we can rewrite (36) and (37) as

$$h(\omega_p) = 1 - \frac{2}{3m} \frac{g^2}{4\pi} \frac{1}{\pi} \int \frac{k^2}{\omega^2} v(k)^2 k\omega \, d\omega \frac{|h(\omega)|^2}{\omega^2-\omega_p^2-i\epsilon}, \tag{39}$$

where the approximation $E_k \simeq \omega_k$ has been used. The solution of (39) is, as easily seen,

$$h(\omega_p) = \frac{1}{1 - \dfrac{g^2}{4\pi} \dfrac{2}{3\pi m} \displaystyle\int \frac{k^3}{\omega} v(k)^2 d\omega \frac{1}{\omega_p^2-\omega^2+i\epsilon}} \tag{40}$$

and the P-wave phase shift δ_p is

$$\tan\delta_p = -\frac{g^2}{4\pi} \frac{p^3}{\omega_p} \frac{v(p)^2}{3m} \frac{1}{1 - \dfrac{g^2}{4\pi} \dfrac{2}{3\pi m} \displaystyle\int \frac{k^3}{\omega} v(k)^2 d\omega \frac{P}{\omega_p^2-\omega^2}}. \tag{41}$$

The effective mass of nucleon is also easily calculated. Using (29), (10), (32), (33) and the approximate expression

$$T_k(q,\ l) \simeq T_k^s(q,\ l) \simeq \delta_{q,1} V_k,$$

which is obtained from (14) in the lowest order approximation, we have

$$(\Psi_q^{(-)} H' \Psi_0) = -\frac{\boldsymbol{P}\cdot\boldsymbol{q}}{m}\frac{V_q}{E_q} - \sum_k \frac{\boldsymbol{P}\cdot\boldsymbol{k}}{m}\frac{V_k}{E_k}\frac{T_k^r(q)}{E_q - E_k + i\epsilon} + \sum_k \frac{\boldsymbol{P}\cdot\boldsymbol{k}}{m}\frac{V_k}{E_q + E_k}\frac{T_k^r(q)}{E_k}. \tag{42}$$

Substituting (38) in (42), we obtain

$$(\Psi_q^{(-)} H' \Psi_0) = -\frac{\boldsymbol{P}\cdot\boldsymbol{q}}{m}\frac{V_q}{E_q}\left(1 + \frac{g^2}{4\pi}\frac{2}{3\pi m}\int \frac{k^3}{\omega} v(k)^2 d\omega \frac{1}{\omega_q^2 - \omega^2 + i\epsilon} h(\omega_q)\right)$$

$$= -\frac{\boldsymbol{P}\cdot\boldsymbol{q}}{m}\frac{V_q}{E_q} h(\omega_q). \tag{43}$$

Hence the perturbed energy ΔE can be written as

$$\Delta E = -\sum_q \frac{(\boldsymbol{P}\cdot\boldsymbol{q})}{m^2}\frac{V_q^2}{\omega_q^3}|h(\omega_q)|^2$$

$$= \frac{\boldsymbol{P}^2}{\pi m}\int d\omega_q [Im\, h(\omega_q)]\frac{1}{\omega_q}, \tag{44}$$

where *Im* means the imaginary part. The integral is easily evaluated by making use of the contour integral used by Lehmann et al.[14] with the result

$$\Delta E = \frac{\boldsymbol{P}^2}{2m}\left\{\frac{1}{1 + \frac{g^2}{4\pi}\frac{2}{3\pi m}\int \frac{k^3}{\omega^3} v(k)^2 d\omega} - 1\right\}. \tag{45}$$

Thus, we obtain the effective mass of nucleon,

$$m^* = m\left\{1 + \frac{g^2}{4\pi}\frac{2}{3\pi m}\int \frac{k}{\omega} v(k)^2 d\omega\right\}. \tag{46}$$

Rewriting the phase shift of (41) in terms of m^*, we have

$$\tan\delta_p = -\frac{g^2}{4\pi}\frac{q^3}{\omega_q^2}\frac{v(q)^2}{3m^*}\frac{1}{1 - \frac{g^2}{4\pi}\frac{2}{3\pi m^*}\omega_q^2 \int \frac{k^3}{\omega^3} v(k)^2 \frac{d\omega P}{\omega_q^2 - \omega^2}} \tag{47}$$

which is just the result obtained by McVoy and Steinwedel.[7]

As the result of these calculations, we now see that the dipole approximation is equivalent to the one-meson approximation. Namely, in making such an approximation, we have taken account of all the effects which may be interpreted as the successions of simple scatterings. For example, the diagrams like (a), (b), (c) in Fig. 1 are all calculated in (47), whereas, regarding the diagram like (d), only the part interpreted like (d_1) is calculated but the part interpreted like (d_2) is not.

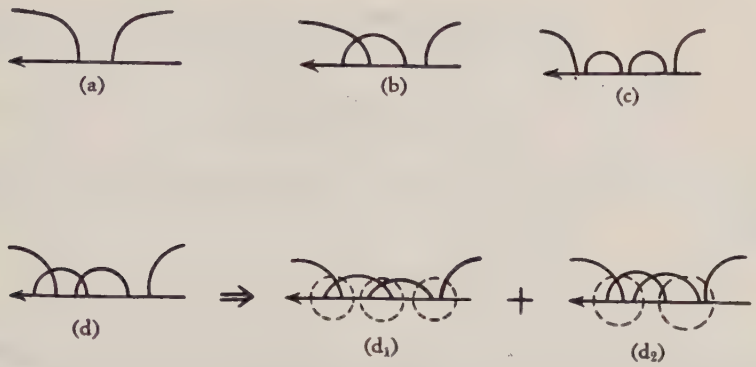


Fig. 1. Diagrams (a), (b) and (c) are regarded as the successions of simple scatterings, while diagram (d) is regarded as either the successions of simple scatterings or the combination of two-meson processes.

B. Effective mass of a polaron

As another example, we here calculate the effective mass of a polaron. Formerly, Lee, Low and Pines[9] calculated it by making use of the intermediate coupling approximation of Tomonaga. However, since the approximation method contains a variation calculation, we cannot interpret the meaning of approximation Feynman-graphically. So we recalculate it by using one-meson approximation and clarify the graphical meaning of the approximation.

The Hamiltonian for the polaron problem is quite similar to that for the neutral scalar meson field.[9] Only differences consist in that ω is the constant frequency of the longitudinal optical mode of the lattice vibrations and

$$V_k=-g\frac{\omega i}{k}\left(\frac{8\pi^2}{m\omega}\right)^{1/4},\quad g^2=\frac{e^2}{2}\left(\frac{2m}{\omega}\right)^{1/2}\left(\frac{1}{n^2}-\frac{1}{\epsilon}\right), \tag{48}$$

where n and ϵ are the optical index of refraction and the static dielectric constant, respectively.

The solutions of the Chew-Low equations for this case are easily obtained, with results

$$\begin{aligned} &T_q^s(0)=V_q,\quad T_q^s(n)=0\quad (n\neq 0),\\ &T_q^r(0)=0,\quad T_q^r(p)=\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m}\frac{V_p^+V_q}{E_pE_q}h(E_p), \end{aligned} \tag{49}$$

where

$$h(E_p)=\frac{1}{1-\sum_k\dfrac{2k^2|V_k|^2}{3mE_k}\cdot\dfrac{1}{E_p^2-E_k^2+i\epsilon}}. \tag{50}$$

(The approximation $E_p\simeq\omega_p$ has not been used in (49).) Substituting (49) in the formulas in Sec. 2–D, we have

$$\Delta E = -\frac{\boldsymbol{P}^2}{2m}\sum_k \frac{2k^2|V_k|^2}{3mE_k^3}|h(E_k)|^2$$

$$= -\frac{\boldsymbol{P}^2}{2m}\frac{2}{\pi}\int \frac{dE_k}{E_k} Im(h(E_k)). \tag{51}$$

The integral may be evaluated in the similar manner as in the neutral scalar meson case and we obtain

$$m^* = m\left(1+\frac{g^2}{6}\right). \tag{52}$$

This result is identical with the value obtained by Lee, Low and Pines, and we see that the Tomonaga approximation is quite equivalent to the one-meson approximation for this case.

C. P-wave scattering of a charged scalar meson

By the above two examples, the validity of our simple approximation method has been well confirmed. Now we proceed to the study of two new problems. The first of them is the *P*-wave scattering of charged scalar meson.

The interaction Hamiltonian for this case is taken to be

$$\boldsymbol{H}_I(\boldsymbol{r}) = \sum_k (V_k a_k e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + V_k^+ a_k^+ e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}),$$

$$V_k = g\frac{v(k)}{\sqrt{2\omega_k}}\tau_k, \tag{53}$$

where τ_k is the k-th component of nucleon isotopic spin operator. (For simplicity, we describe the momentum and isotopic spin of meson by a single symbol k.)

In this case, we cannot solve the Chew-Low equations even in the static approximation. But the solution in the one-meson approximation[7] serves our purpose, if the coupling constant is not so large. Accordingly,

$$T_q^s(0) = V_q$$

$$T_q^s(p) = -(4\pi v(p)v(q)/\sqrt{4\omega_p\omega_q})\sum_{\alpha=0,1} b_\alpha(\omega_p) I_\alpha(p, q) \tag{54}$$

$$T_q^s(p_1, p_2) = 0, \quad \text{etc.},$$

shall be used below as the static scattering amplitudes. In (54),

$$I_0(p, q) = \delta_{pq} - \frac{1}{2}\tau_p\tau_q \quad \text{and} \quad I_1(p, q) = \frac{1}{2}\tau_p\tau_q$$

are the projection operators which extract the positive and negative meson scattering by a proton, respectively, and

$$b_\alpha(\omega_p) = \frac{\Lambda_\alpha}{\omega_p}\left[1 - \Lambda_\alpha\frac{2\omega_p}{\pi}\int_u^\infty d\omega_k \frac{kv(k)^2}{\omega_k}\frac{1}{\omega_k^2-\omega_p^2-i\epsilon}\right]^{-1}$$

$$\to \frac{\Lambda_\alpha}{\omega_p - \Lambda_\alpha(\mu+ip)} \quad (\text{in the limit } v(k)\to 1), \tag{55}$$

where $\Lambda_\alpha=(-)^\alpha\Lambda=(-)^\alpha 2(g^2/4\pi)$.

Substituting (54) and (55) in (26), we easily obtain

$$(\mathcal{F}_0 w^r_{p'q} \mathcal{F}_0)=\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m}4\pi\frac{v(p)v(q)}{\sqrt{4\omega_p\omega_q}}\frac{1}{\mu^2}\sum_\alpha I_\alpha(p,\ q)W_\alpha(\omega_p,\ \omega_q), \tag{56}$$

$$W_\alpha(\omega_p,\ \omega_q)=\frac{\Lambda}{\mathcal{E}\mathcal{E}'}\delta_{\alpha 0}+\frac{\Lambda_\alpha\mathcal{E}_\alpha}{2\pi}\left[\frac{1}{\mathcal{E}'-\mathcal{E}}\left\{\frac{\sqrt{\mathcal{E}^2-1}}{\mathcal{E}(\mathcal{E}-\mathcal{E}_\alpha)}\cosh^{-1}\mathcal{E}\right.\right.$$

$$\left.\left.-\frac{\sqrt{\mathcal{E}'^2-1}}{\mathcal{E}'(\mathcal{E}'-\mathcal{E}_\alpha)}\cosh^{-1}\mathcal{E}'\right\}-\frac{\pi}{2\mathcal{E}_\alpha\mathcal{E}\mathcal{E}'}+\frac{\sqrt{1-\mathcal{E}_\alpha^2}}{\mathcal{E}_\alpha(\mathcal{E}-\mathcal{E}_\alpha)(\mathcal{E}'-\mathcal{E}_\alpha)}\cos^{-1}\mathcal{E}_\alpha\right], \tag{57}$$

where $\mathcal{E}=\omega_p/\mu$, $\mathcal{E}'=\omega_q/\mu$, $\mathcal{E}_\alpha=2\Lambda_\alpha/(1+\Lambda^2)$ and we have neglected the terms not contributing to P-wave scattering. Similarly, $(\mathcal{F}_0\tilde{w}^r_{p'q}\mathcal{F}_0)$ is obtained by replacing $W_\alpha(\omega_p,\ \omega_q)$ in (56) by

$$\widetilde{W}_\alpha(\omega_p,\ \omega_q)=\frac{\Lambda}{\mathcal{E}\mathcal{E}'}\delta_{\alpha 0}-\frac{\mathcal{E}_\alpha(\mathcal{E}'+\Lambda_\alpha)}{2\mathcal{E}'(\mathcal{E}'+\mathcal{E}_\alpha)(\mathcal{E}+\mathcal{E}')}$$

$$+\frac{\Lambda_\alpha\mathcal{E}_\alpha}{2\pi}\left[\frac{1}{\mathcal{E}+\mathcal{E}'}\left\{\frac{\sqrt{\mathcal{E}^2-1}}{\mathcal{E}(\mathcal{E}-\mathcal{E}_\alpha)}\cosh^{-1}\mathcal{E}+\frac{\sqrt{\mathcal{E}'^2-1}}{\mathcal{E}'(\mathcal{E}'+\mathcal{E}_\alpha)}\cosh^{-1}\mathcal{E}'\right\}\right.$$

$$\left.-\frac{\pi}{2\mathcal{E}_\alpha\mathcal{E}\mathcal{E}'}+\frac{\sqrt{1-\mathcal{E}_\alpha^2}}{\mathcal{E}_\alpha(\mathcal{E}-\mathcal{E}_\alpha)(\mathcal{E}'+\mathcal{E}_\alpha)}\cos^{-1}\mathcal{E}_\alpha\right]. \tag{58}$$

Then, setting

$$\begin{Bmatrix}T^r_q(p)\\ S^r_q(p)\end{Bmatrix}=\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m}4\pi\frac{v(p)v(q)}{\sqrt{4\omega_p\omega_q}}\frac{1}{\omega_p\omega_q}\sum_\alpha I_\alpha(p,\ q)\begin{Bmatrix}f_\alpha(\omega_p)\\ g_\alpha(\omega_p)\end{Bmatrix} \tag{59}$$

and substituting all of these in (22) and (23), we have the equations for the recoil parts in the one-meson approximation,

$$f_\alpha(\omega_p)=\mathcal{E}\mathcal{E}'W_\alpha(\omega_p,\ \omega_q)-\frac{1}{3\pi m}\int d\omega\,\frac{k^3v(k)^2}{\omega^2}\left[\frac{|f_\alpha(\omega)|^2}{\omega-\omega_p-i\epsilon}+\frac{|g_\alpha(\omega)|^2}{\omega+\omega_p}\right],$$

$$g_\alpha(\omega_p)=\mathcal{E}\mathcal{E}'\widetilde{W}_\alpha(\omega_p,\ \omega_q)-\frac{1}{3\pi m}\int d\omega\,\frac{k^3v(k)^2}{\omega^2}\left[\frac{f^+_\alpha(\omega)g_\alpha(\omega)}{\omega-\omega_p-i\epsilon}+\frac{f^+_\beta(\omega)g_\beta(\omega)}{\omega+\omega_p}\right], \tag{60}$$

where $\beta=1-\alpha$.

These equations, however, cannot be solved because of the complicatedness of W_α and $\widetilde{W}_\alpha$. Thereupon we first attempt to solve the approximate equations, assuming

$$W_\alpha(\omega_p,\ \omega_q)\simeq\frac{\Lambda}{\mathcal{E}\mathcal{E}'}\delta_{\alpha 0},\quad \widetilde{W}_\alpha(\omega_p,\ \omega_q)\simeq\frac{\Lambda}{2\mathcal{E}\mathcal{E}'}. \tag{61}$$

This approximation holds good numerically except for the case of the very large values of $\mathcal{E}$, $\mathcal{E}'$ and Λ.

Making such a simplification, we can easily solve the equations, and the solutions are

$$f_0(\omega_p) = \frac{\Lambda + (\Lambda^2/4)\eta_f}{1 - \Lambda\tilde{\xi} - (\Lambda^2/4)\tilde{\xi}\eta_f},$$

$$f_1(\omega_p) = \frac{(\Lambda^2/4)\eta_f}{1 - \Lambda\eta_f - (\Lambda^2/4)\tilde{\xi}\eta_f},$$

$$g_0(\omega_p) = \frac{(\Lambda/2)}{1 - \Lambda\tilde{\xi} - (\Lambda^2/4)\tilde{\xi}\eta_f},$$

$$g_1(\omega_p) = \frac{(\Lambda/2)}{1 - \Lambda\eta_f - (\Lambda^2/4)\tilde{\xi}\eta_f}, \tag{62}$$

where

$$\begin{pmatrix}\tilde{\xi}\\ \eta_f\end{pmatrix} = -\frac{1}{3\pi m}\int_{\mu}^{\infty} d\omega \frac{k^3 v(k)^2}{\omega^2}\,\frac{1}{\omega_k \mp \omega_p - i\epsilon}. \tag{63}$$

Numerically examining these solutions, we see that the second terms of (60) is much smaller than the neglected term in (61) for the cut-off of the order of nucleon mass. Hence, we can see that, in order to obtain the closely approximate values of $T_q^r(p)$, we may rather neglect the second terms of (60) but should use the full expression (57) for W_α. Physically, this means that the effects of the virtually propagating P-wave mesons (c. f. Fig. 2–a) are small but those of virtually propagating S-wave mesons (c. f. Fig. 2–b) are so important that we must fully take account of them.

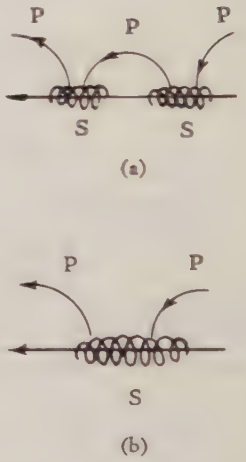


Fig. 2. The effect of (a) is much smaller than that of (b). (In the figure, the coil-like lines mean the S-wave meson cloud.)

Thus, we finally have

$$\delta_\alpha^P \simeq -\frac{f^2}{3m}\sqrt{\mathcal{E}^2-1}^3\left[\Lambda_\alpha\delta_{\alpha 0} + \frac{\Lambda_\alpha\mathcal{E}_\alpha}{2\pi}\left\{-\frac{\cosh^{-1}\mathcal{E}}{(\mathcal{E}-\mathcal{E}_\alpha)\sqrt{\mathcal{E}^2-1}}\right.\right.$$
$$\left.\left. + \frac{\mathcal{E}\sqrt{\mathcal{E}^2-1}}{(\mathcal{E}-\mathcal{E}_\alpha)^2}\cosh^{-1}\mathcal{E} - \frac{\mathcal{E}}{\mathcal{E}-\mathcal{E}_\alpha} - \frac{\pi}{2\mathcal{E}_\alpha} + \frac{\sqrt{1-\mathcal{E}_\alpha^2}\mathcal{E}^2}{\mathcal{E}_\alpha(\mathcal{E}-\mathcal{E}_\alpha)^2}\cos^{-1}\mathcal{E}_\alpha\right\}\right] \tag{64}$$

as the P-wave phase shifts due to the nucleon recoil, where the relation

$$f_\alpha(\omega_p) = -\frac{3m\omega_p^2}{p^3}\sin\delta_\alpha^P e^{i\delta_\alpha^P} \simeq -\frac{3m\omega_p^2}{p^3}\delta_\alpha^P \tag{65}$$

has been used.

The discussion on the results will be made in the last section, together with the results of the next subsection.

D. P-wave scattering in meson pair theory

Another problem we must discuss is the P-wave scattering in the meson pair theory

of Wentzel.[11] In this theory, the interaction Hamiltonian is

$$\boldsymbol{H}_I(\boldsymbol{r}) = \frac{4\pi\lambda_0}{2\mu}\left[\sum V_k (a_k e^{i\boldsymbol{k}\cdot\boldsymbol{r}} + a_k^+ e^{-i\boldsymbol{k}\cdot\boldsymbol{r}})\right]^2 \tag{66}$$

where $V_k = v(k)/\sqrt{2\omega_k}$ and λ_0 is the unrenormalized coupling constant. For this case, we can easily solve the static Chew-Low equation, with the results

$$T_q^s(p) = \frac{4\pi\lambda_0}{\mu} V_p V_q h_0(\omega_p),$$

$$h_0(\omega_p) = \frac{1}{1 - \dfrac{2\lambda_0}{\mu\pi}\displaystyle\int d\omega\, k\omega\, v(k)^2 \frac{1}{\omega_p^2 - \omega^2 + i\epsilon}}. \tag{67}$$

(All other amplitudes vanish.) In terms of the 'renormalized' coupling constant

$$\lambda = \lambda_0\left[1 + \frac{2\lambda_0}{\mu\pi}\int d\omega \frac{k v(k)^2}{\omega}\right]^{-1},$$

we may write (67) as

$$T_q^s(p) = \frac{4\pi\lambda}{\mu} V_p V_q h(\omega_p),$$

$$h(\omega_p) \to 1/[1 + \lambda + i\lambda(p/\mu)] \quad \text{(in the limit } v(k) \to 1). \tag{68}$$

Substituting this in (26), we have

$$\begin{aligned}(\mathfrak{F}_0 w_{pq}^r \mathfrak{F}_0) = \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{m}\,\frac{4V_p V_q}{\mu^2}\Bigg[&\frac{1}{\mathcal{E}'-\mathcal{E}}\left\{\frac{\sqrt{\mathcal{E}^2-1}}{\mathcal{E}^2+\mathcal{E}_0^2}\cosh^{-1}\mathcal{E}\right.\\ &\left.-\frac{\sqrt{\mathcal{E}'^2-1}}{\mathcal{E}'^2+\mathcal{E}_0^2}\cosh^{-1}\mathcal{E}'\right\} + \frac{\pi}{2}\,\frac{(\mathcal{E}+\mathcal{E}')\sqrt{\mathcal{E}_0^2+1}}{(\mathcal{E}^2+\mathcal{E}_0^2)(\mathcal{E}'^2+\mathcal{E}_0^2)}\\ &-\frac{\mathcal{E}\mathcal{E}'-\mathcal{E}_0^2}{(\mathcal{E}^2+\mathcal{E}_0^2)(\mathcal{E}'^2+\mathcal{E}_0^2)}\,\frac{\sqrt{\mathcal{E}_0^2+1}}{\mathcal{E}_0}\sinh^{-1}\mathcal{E}_0\Bigg],\end{aligned} \tag{69}$$

where $\mathcal{E} = E_p/\mu$, $\mathcal{E}' = E_q/\mu$, $\mathcal{E}_0 = \sqrt{1+2\lambda}/\lambda$ and we have neglected the terms which do not contribute to P-wave scatterings.

If the same approximation as the charged scalar case holds good, we may regard this as $T_q^r(p)$ itself. Thus, remembering

$$T_q^r(p)\big|_{E_p = E_q} = -\frac{6\pi}{p^3\omega}\boldsymbol{p}\cdot\boldsymbol{q}\sin\delta_p e^{i\delta_p},$$

we have the P-wave phase shift due to nucleon recoil,

$$\begin{aligned}\delta_p \cong -\frac{\mu}{3\pi m}\sqrt{\mathcal{E}^2-1}^3\Bigg[&\frac{2\mathcal{E}\sqrt{\mathcal{E}^2-1}}{(\mathcal{E}^2+\mathcal{E}_0^2)^2}\cosh^{-1}\mathcal{E}\\ &-\frac{\mathcal{E}}{\sqrt{\mathcal{E}^2-1}(\mathcal{E}^2+\mathcal{E}_0^2)}\cosh^{-1}\mathcal{E} - \frac{1}{\mathcal{E}^2+\mathcal{E}_0^2}\end{aligned}$$

$$+\frac{\pi\varepsilon\sqrt{1+\varepsilon_0^2}}{(\varepsilon^2+\varepsilon_0^2)^2}-\frac{(\varepsilon^2-\varepsilon_0^2)\sqrt{1+\varepsilon_0^2}}{(\varepsilon^2+\varepsilon_0^2)^2}\frac{\sinh^{-1}\varepsilon_0}{\varepsilon_0}\Bigg]. \tag{70}$$

§ 4. Summary and Discussions

Here we briefly summarize the main results. In Sec. 2, bearing in mind the tightly interacting systems of a quantized field and a non-relativistically moving particle, we developed a general method which is very convenient for obtaining the recoil correction to scattering amplitudes and the effective mass of particle. The results are as follows: First, making use of (17) and (18), we calculate the Chew-Low amplitudes for the system in question in the static approximation. Then, substituting them in (26) and (27), we calculate $(\Psi_0 w_{pq} \Psi_0)$ and $(\Psi_0 \tilde{w}_{pq} \Psi_0)$. Using these in (22) and (23), we make the equations for $T_\mu^\nu(m)$ and $S_\mu^\nu(m)$, of which the solutions are just the recoil parts of scattering amplitudes we are seeking. Also the effective mass of the particle is evaluated by making use of (28) and (29), in which $T_3(n, \varepsilon)$ is obtained by solving (14) and (15).

In Sec. 3, we applied this method to four problems. Because we laid stress upon the methodological points, the examples were necessarily confined to the simplest ones. But, even from these examples, we can deduce many interesting conclusions. From the first two examples, we find that the dipole approximation for the scattering of neutral scalar mesons and the Tomonaga approximation for the polaron problem are equivalent to the one-meson approximation.

From the calculation of the P-wave scattering of charged scalar meson, we may perceive the following interesting facts: (i) In the lowest-order perturbation theory, there occurs the P-wave scattering in the case of $\pi^- + p \to \pi^- + p$ for which, Feynman-graphically speaking, the meson lines cross (Fig. 3 a), while it does not in the case of $\pi^+ + p \to \pi^+ + p$ for which the meson lines do not cross (Fig. 3 b). However, in the present results, the P-wave scattering occurs even in the case of the negative meson scattering. It is because, in our method, the higher-order processes, like Fig. 3 c for example, are also taken into account. (ii) Moreover, when the meson energy is high to some degree, both P-wave phase shifts for the π^+ and π^- scatterings are of the same order and their energy dependences are of the form $(\ln\varepsilon)\,\varepsilon$. (Because the present method is non-relativistic, the result of (64) cannot be used in very high energy cases. But, note that the term proportional to $(\varepsilon^2-1)^{\frac{1}{2}}\cosh^{-1}\varepsilon\,\varepsilon(\varepsilon-\varepsilon_1)^2$ contributes mainly even for $\omega \sim 7\mu$ except for the case of very small coupling constant.) (iii) The effect due to the virtual P-wave mesons is much smaller than that of virtual S-wave mesons.

In the case of pair meson theory, we can perceive the circumstances similar to the charged scalar case as to the contribution of virtual P-wave mesons, the energy dependence of P-wave phase shift, etc. But it is the most interesting that the phase shift depends on the coupling constant in the form $\lambda^2 \ln\lambda$ rather than λ^2 when λ is very small. All these facts cannot be understood from the standpoint of perturbation theory.

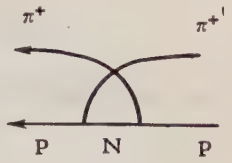
Of course, in order to insist the correctness of these strange results strongly, we

must more closely examine the validity of our approximations. Indeed, the two meson effects are not necessarily small for some cases.[15] It is possible that the strangeness of the results are due to the one-meson approximation itself. However, we do not go into this problem, because the one-meson approximation holds good for many cases.

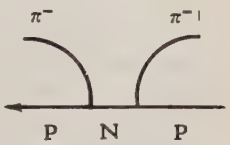
Thus we have been able to realize the importance of the non-relativistic recoil effects, specially of taking account of them to the higher order of coupling constant.

Though no realistic problems have been discussed here, it is possible that this effect is similarly important for those cases. Particularly, they may contribute largely to the phenomena for which the nucleon recoil plays an essential role even in the perturbation theory, for example, to the anomalous magnetic moment of nucleon, to the Compton scattering by nucleon, etc. Close investigations on these problems are strongly desired.

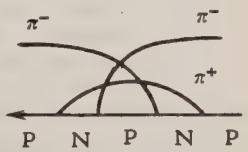
The authors wish to express their sincere thanks to Professor H. Fukuda.



(a)



(b)



(c)

Fig. 3. For the case of charged scalar theory, the lowest-order *P*-wave scattering of positive mesons (diagram (a)) occurs, but that of negative ones (diagram (b)) does not. However, in the present method, the negative *P*-wave meson scattering also occurs, because the diagrams like (c) are taken into account. (In the figure, *P* and *N* denote a proton and a neutron, respectively.)

Appendix

Here we prove the formula (13) briefly. Using eq. (6), we have

$$N_n a_{p_1}^{+} \cdots a_{p_n}^{+} \Psi_s^{(-)} = N_n a_{p_1}^{+} \cdots a_{p_n}^{+} \left\{ N_s a_{q_1}^{+} \cdots a_{q_s}^{+} \Psi_0 - \frac{1}{H - E_s + i\epsilon} U_s \Psi_0 \right\}$$

$$= \frac{N_n N_s}{N_{n+s}} \left\{ \Psi_{n+s}^{-} + \frac{1}{H - E_{n+s} + i\epsilon} U_{n+s} \Psi_0 \right\} - N_n a_{p_1}^{+} \cdots a_{p_n}^{+} \frac{1}{H - E_s + i\epsilon} U_s \Psi_0 . \qquad (A \cdot 1)$$

Remembering

$$N_n a_{p_1}^{+} \cdots a_{p_n}^{+} \frac{1}{H - E_s + i\epsilon} = N_n \frac{1}{H - E_{n+s} + i\epsilon} a_{p_1}^{+} \cdots a_{p_n}^{+} + \frac{1}{H - E_{n+s} + i\epsilon} U_n \frac{1}{H - E_s + i\epsilon} ,$$

we may write (A·1) as

$$N_n a_{p_1}^{+} \cdots a_{p_n}^{+} \Psi_s^{(-)} = \frac{N_n N_s}{N_{n+s}} \Psi_{s+n}^{(-)} + \frac{1}{H - E_{n+s} + i\epsilon} \left\{ \frac{N_n N_s}{N_{n+s}} U_{n+s} \right.$$

$$-N_n a_{p_1}^{+} \cdots a_{p_n}^{+} U_s \Big\} \Psi_0 - \frac{1}{H-E_{n+s}+i\epsilon} U_n \frac{1}{H-E_s+i\epsilon} U_s \Psi_0 . \qquad (A\cdot 2)$$

Now, substituting the relation

$$\frac{N_n N_s}{N_{n+s}} U_{n+s} - N_n a_{p_1}^{+} \cdots a_{p_1}^{+} U_s = N_s U_n a_{q_1}^{-} \cdots a_{q_s}^{-}$$

in the second term of (A·2), we obtain

$$N_n a_{p_1}^{+} \cdots a_{p_n}^{+} \Psi_s^{(-)} = \frac{N_n N_s}{N_{n+s}} \Psi_{s+n}^{(-)} + \frac{1}{H-E_{n+s}+i\epsilon} U_n \Big\{ N_s a_{q_1}^{-} \cdots a_{q_s}^{-} \Psi_0$$

$$- \frac{1}{H-E_s+i\epsilon} U_s \Psi_0 \Big\} ,$$

that is,

$$N_n a_{p_1}^{+} \cdots a_{p_n}^{+} \Psi_s^{(-)} = \frac{N_n N_s}{N_{n+s}} \Psi_{n+s}^{(-)} + \frac{1}{H-E_{n+s}+i\epsilon} U_n \Psi_s^{(-)} . \qquad (A\cdot 3)$$

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Tensor and Spin-Orbit Forces in Nucleon-Nucleon Scattering*

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In order to investigate effects of a tensor force and to examine necessity of introducing a spin-orbit force at high energies, nucleon-nucleon scattering at 150 Mev is analysed. The analysis is made taking into account some characteristic features of the pion-theoretical nuclear forces. It is shown that the experimental data around 150 Mev can be well explained by two main features of nuclear forces; (i) a strong tensor force in the outer part of the interaction and (ii) a hard-core-like repulsive interaction in the inner part. It should be emphasized that the former feature, which is the most characteristic one of the pion-theoretical nuclear forces, is decisively important at such high energies as well as at low energies. We find no positive evidences for such spin-orbit forces as play an important role around 150 Mev. This conclusion is in conflict with the prediction of very strong spin-orbit forces recently made by Signell and Marshak and by Gammel and Thaler. This point is discussed quite in detail. Especially, it is shown that Signell and Marshak's spin-orbit coupling potential comes from their undue reliance on the inner part of Gartenhaus' potential.

§ 1. Introduction

As is well known, the static pion theory has made remarkable success for the low energy two-nucleon system.** [1] In addition, proton-proton scattering at 90 Mev was analysed by Otsuki[2] and neutron-proton scattering at the same energy by Watari,[3] from the pion-theoretical point of view. They used the pion-theoretical potential in the outer region and treated the nuclear interaction in the inner region phenomenologically. Their results showed that the nucleon-nucleon scattering data at about 90 Mev can be completely reproduced by the static pion-theoretical potential established at the low energy regions. Especially, it has been made clear that the very strong tensor force due to the one-pion-exchange process, which is the most characteristic feature of the pion-theoretical potential in the outer region, is of decisive importance.

The success of these attempts is one of the two motivations of the present work. Although the successful method of approach which we have hitherto adopted from the standpoint of the pion theory is no longer valid above about 100 Mev for many reasons as stated in ref. 1) and § 2, it is expected that some characteristic features of the low energy pion theory play also an important role even above 100 Mev. In particular,

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* Some preliminary report of this work has already been published in this journal.[2,3]

** A full report on the verification of the low energy pion theory of nuclear forces has been published in Supplement of this journal (No. 3, 1956).[1] Notations and terminologies used in this paper are the same with those in it.

consequences of the strong tensor force as predicted by the low energy pion theory are to be investigated.

The other motivation of the present work is the introduction of very strong spin-orbit coupling potentials recently made by Signell and Marshak[4] and by Gammel and Thaler.[5] According to their analyses, a strong spin-orbit coupling potential seems to be favorable in explaining high energy data. However, their spin-orbit coupling potentials have no theoretically sound basis. Moreover, they are unreasonably strong in the outer region compared with many pion-theoretical results derived by various methods.[6,7]

In the present paper, a phenomenological analysis at 150 Mev is made in close contact with the characteristic features of the pion-theoretical nuclear forces. It is investigated whether a strong tensor force in the outer part of the interaction plays an essential role as at the lower energies and what natures are neccesary in the inner part for the explanation of the experimental data. The reasons why we should choose this energy are as follows: (i) We have plenty of useful experimental data. (ii) The P-wave impact parameter b_1 is $0.7(\hbar/\mu c)$. So that, above 150 Mev the inner part of the nuclear interaction generally becomes more and more important to the P-wave phase shifts than the outer part does. (iii) It is interesting to compare our results with Signell and Marshak's because their spin-orbit coupling potential was introduced mainly by their analysis around 150 Mev.

Our results show that the high energy scattering data around 150 Mev can be explained if the nuclear interaction has the following two features: a strong tensor force in the outer region which has the same exchange character as the pion theoretical potential and a hard-core-like repulsive interaction in the inner region. We can find no evidences for any strong spin-orbit coupling forces as introduced by Signell and Marshak and by Gammel and Thaler. It will be discussed in § 6 that the former spin-orbit term was introduced only to eliminate undesirable effects due to the inner part of Gartenhaus' potential on which their analysis was based. If the hard-core cut-off procedure is adopted instead of their zero cut-off, such an unreasonably strong and long range spin-orbit potential is not neccesary. On the other hand, Gammel and Thaler's potential is one of the possible ways to explain the scattering data at about 150 Mev. However, their strong spin-orbit potential is hardly accepted, since it destroys the characteristic features of the pion theory even at low energies.

In § 2, the method of approach employed in the present paper will be discussed, and qualitative features of the scattering parameters (phase shifts and mixing ratios), which are taken into consideration throughout this work, will be explained. In terms of these scattering parameters, we shall analyse in detail proton-proton scattering in § 3 and then neutron-proton scattering in § 4. In § 5, we shall summarize the properties of the nuclear interaction indicated by the characteristic features of the scattering parameters determined in § 3 and 4. Our conclusion on the spin-orbit coupling potential will be compared with those of Signell and Marshak and Gammel and Thaler in § 6 and the formers are criticized in detail. § 7 will be devoted to conclusions.

§ 2. Preliminary discussions

A. *Method of approach*

As stated in ref. 1), applicability of the successful method of analysis which we have hitherto adopted from the standpoint of the pion theory is considered to be limited to rather lower energies ($E_{lab} < 100$ Mev). The reasons are as follows: When the energy of the two-nucleon system becomes higher than about 100 Mev, the *P*-wave phase shifts, which are the most important quantities in determining both polarized and unpolarized cross sections, become to be more and more seriously affected by the inner part* of the interaction. However, the present day quantum theory cannot give any reliable predictions on the inner interaction. Furthermore, even for the outer part,* dynamical relativistic corrections to the one- and two-pion-exchange potentials may be appreciable and modify them in a considerable extent. Also, the kinematical relativistic effects for the treatment of the two-nucleon system become not negligible. Due to such various factors, most of which are unknown, the method of approach above 100 Mev should be far more phenomenological than that at lower energies.

Therefore, a direct application of the static pion-theoretical potential to the high energy scatterings[8] is not appropriate. In such an approach, it is difficult to discriminate the effects due to the characteristic features of the low energy pion theory which still remain unchanged, from the effects newly revealed at high energies but almost hidden at low energies. On the other hand, a purely phenomenological method such as the so-called phase shift analysis[9] gives no valuable information, because due consideration is not given to the characteristic features of the pion-theoretical nuclear forces.

The method of analysis employed in the present paper is phenomenological and may be regarded as a kind of phase shift analysis. However, contrary to the usual phase shift analysis it is based on the assumption that a tensor potential is strongest of all nuclear forces, corresponding to the most characteristic feature of the pion-theoretical nuclear forces. This tensor potential has the same exchange character with the one-pion-exchange potential:

$$V^{(1)}(x) = \frac{1}{3} \cdot \frac{g_e^2}{4\pi} \cdot \mu c^2 (\tau_1 \cdot \tau_2) \left\{ (\sigma_1 \cdot \sigma_2) + S_{12} \left(1 + \frac{3}{x} + \frac{3}{x^2} \right) \right\} \frac{e^{-x}}{x}. \qquad (2 \cdot 1)$$

First of all, we investigate whether some sets of such "tensor-dominant type" phase shifts can explain the qualitative aspects of the data at 150 Mev. If we cannot find such a set, it means that new features of nuclear forces, that are hidden in the low energy phenomena, become important as the energy goes high. Concerning the inner part of the interaction, a hard-core-like repulsive interaction is expected as the most characteristic feature. In the present paper, however, we *do not assume its existence a priori*. Rather, we shall try to find positive evidences for its existence by comparison with the data.

* As used in ref. 1), the outer part means the region $x \geq 0.7$ and the inner part the region $x \leq 0.7$, where x is the inter-nucleon distance in unit of the pion compton wave length $\hbar/\mu c = 1{,}415 \times 10^{-13}$ cm.

B. *Qualitative features of scattering parameters*

By the impact parameter consideration,[1] we can divide the scattering parameters into three groups by discriminating what part of the interaction mainly determine their qualitative features; (i) the scattering parameters of the higher waves with $L \geq 2$* which are mainly affected by the outer part of the interaction, (ii) the P-wave scattering parameters, whose characteristic features are determined by the outer part but considerably affected by the inner part, and (iii) the S-wave scattering parameters which depend seriously on the inner part and should be treated in a purely phenomenological way. Based on these general criterions, we summarize the qualitative features of scattering parameters.

(1) *Uncoupled higher waves* ($L \geq 2$) : The impact parameters of the F-and D-wave are 1.7 and 1.3, respectively. Therefore, we fix the phase shifts of these waves in the pion-theoretical values ($f_e^2/4\pi = 0.08$). Their values are given in Table 1. For scattering parameters of coupled higher waves a somewhat phenomenological treatment is employed as will be discussed in (3) and (5). Also, it is to be noted that the ${}^3D_2$-phase shifts, which are large due to the strong tensor force of the one-pion-exchange potential, have some ambiguity owing to the uncertainty of the central part of the two-pion-exchange potential in the triplet even state. So, we take the values, $\delta_2^3 = 0.45 \pm 0.10$. The scattering parameters with $J \geq 4$ are safely neglected at 150 Mev.

(2) ${}^3P_J$*-state* ($J=0, 1, 2$) : We assume the ${}^3P$ -phase shifts of the tensor dominant type. Since the tensor potential is positive in the triplet odd state, we have

$$\delta_0^T \gg 0, \quad \delta_1^3 < 0, \quad \delta_2^\alpha > 0. \tag{2·2}$$

Furthermore, if we divide δ into two parts as

$$\left.\begin{aligned} \delta_0^T &\equiv \delta_0^{T(1)} + \Delta_0^T, \\ \delta_1^3 &\equiv \delta_1^{3(1)} + \Delta_1^3, \\ \delta_2^\alpha &\equiv \delta_2^{\alpha(1)} + \Delta_2^\alpha, \end{aligned}\right\} \tag{2·3}$$

we have

$$\delta_0^{T(1)} : \delta_1^{3(1)} : \delta_2^{\alpha(1)} \simeq 4 : -2 : r \,(0 < r < 1). \tag{2·4}$$

Here $\delta^{(1)}$ means the contribution from the one-pion-exchange potential. (2·4) corresponds to the tensor dominant property of this potential. Δ contains all the other contributions. Of these the most important is the contribution from the two-pion-exchange central potential, whose attractive property has been confirmed by the low energy analysis,[1] and is effectively approximated by a square well potential with the depth 50 Mev and the range 1. The tensor part of the two-pion-exchange potential is very small. Also as will be discussed in § 6, the contribution to Δ with the spin-orbit coupling type should be considered small. Thus, we can approximate

* We denote the orbital angular momentum and the total angular momentum by L and J respectively

$$\Delta_0^{\tau}=\Delta_1^{3}=\Delta_2^{\alpha}\equiv\Delta_c.$$

In the Born approximation, we get $\delta_0^{\tau(1)}\sim 0.5$. $\delta_1^{3(1)}\sim -0.3$, and $\Delta_c\sim 0.2$. However, if we assume a hard-core-like repulsive interaction, Δ_c is more or less reduced. For $x_c\sim 0.4$, Δ_c is reduced to zero, where x_c is the radius of the infinite repulsive core.

From the above discussions, we can summarize the conditions for the ${}^3P_J$-state as follows:

$$\left.\begin{aligned}&\delta_0^{\tau}\sim 0.5+\Delta_c,\quad \delta_1^{3}\sim -0.3+\Delta_c,\\ &0<\delta_2^{\alpha(1)}\lesssim 0.15,\quad 0<\Delta_c\lesssim 0.2.\end{aligned}\right\}\qquad (2\cdot 5)$$

(3) *Mixing ratio* ϵ_J*: In the cases where tensor forces are very strong, ϵ_J is an important quantity for which careful treatment is required. Both unpolarized and polarized cross sections are sensitive to ϵ_J at high energies, as is seen from Otsuki[2] and Watari's[3] results at 90 Mev. ϵ_J is considerably affected not only by the outer part of the interaction but also by the inner part, so we treat them phenomenologically, as will be mentioned in (4), (5), and (8).

(4) ${}^3P_2+{}^3F_2$*-state*: The one-pion-exchange potential gives $\epsilon_2<0$ and $0>\delta_2^{\tau(1)}>-\delta_2^{\alpha(1)}$, in the Born approximation. Thus, in the existence of the strong tensor force with the positive sign, it is expected that

$$\epsilon_2<0,\quad 0.15\gtrsim\delta_2^{\alpha(1)}>0\quad\text{and}\quad 0>\delta_2^{\tau}\gtrsim -0.1.\qquad (2\cdot 6)$$

(5) ${}^3D_3+{}^3G_3$*-state*: The scattering parameters δ_3^{2}, δ_3^{τ} and ϵ_3 are delicate quantities because of possible cancellations of terms contributing to them, although they are mainly determined by the outer part of the interaction. Therefore, we set no other conditions than

$$|\delta_3^{2}|,\ |\delta_3^{\tau}|\lesssim 0.1\quad\text{and}\quad |\epsilon_3|<\pi/4.\qquad (2\cdot 7)$$

(6) ${}^1P_1$*-state*: In this state the one-pion-exchange potential gives the most dominant effect due to its large kinematical factor $(\sigma_1\cdot\sigma_2)(\tau_1\cdot\tau_2)=9$. The contribution to the ${}^1P_1$-phase shift ${}^1\delta_1$ from the outer part of this potential is about -0.2. A hard-core-like repulsive interaction gives the contributions of the same order of magnitude. So, we set the condition for ${}^1\delta_1$ as

$${}^1\delta_1=-0.2\sim -0.4.\qquad (2\cdot 8)$$

(7) ${}^1S_0$*-state*: ${}^1\delta_0$ should be treated phenomenologically and is determined from the experimental value of the effective total cross section of $p-p$ scattering $\sigma_{pp}^{\text{eff}}\sim 2\pi(d\sigma(90^\circ)/d\Omega)_{pp}$, when the higher wave ($L\geq 1$) contributions are given. If we take into account the uncertainty of the higher wave contributions, the following condition is adopted in connection with the 90 Mev value[2]:

$$0<{}^1\delta_0<0.4.\qquad (2\cdot 9)$$

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* The definition of ϵ_J used in the present paper is the same as ϵ_J^2 in ref. 1) and ϵ_J in Blatt-Biedenharn's notations.[10]

(8) $^3S_1+{}^3D_1$-*state*: This state should be treated phenomenologically. Conditions for the scattering parameters δ_1^{α}, δ_1^{τ} and ϵ_1 are as follows:

(i) Values of these quantities at 150 Mev should be continued to those at the lower energies ($\delta_1^{\alpha}\sim 0.7$, $\delta_1^{\tau}\sim -0.3$ and $\epsilon_1\sim 0$ at 90 Mev). δ_1^{α} and δ_1^{τ} should be smaller than those at 90 Mev.

(ii) Subtracting the other wave contributions from the experimental value of the $u-p$ total cross section, $\sigma_{np}\cong 50$ mb, we obtain $\sin^2\delta_1^{\alpha}+\sin^2\delta_1^{\tau}\lesssim 0.4$. Therefore we can set the restrictions as follows;

$$0<\delta_1^{\alpha}\lesssim -\delta_1^{\tau}\lesssim 0.4 \quad \text{and} \quad |\epsilon_1|<\pi/4. \tag{2·10}$$

At the end of this section, we summarize the values of the scattering parameters or the conditions for them discussed above, in Table 1.

Table 1. Qualitative features of the scattering parameters at 150 Mev (in unit of radian). Notations of the phase shifts and the mixing ratios are the same as these used in ref. 1).

S / J / T	1 (triplet)				0 (singlet)			
	0		1		1		0	
0			$^3P_0$	$\delta_0^{\tau}\sim -0.5+\Delta_c$	$^1S_0$	$0<{}^1\delta_0<0.4$		
1	$^3S_1+{}^3D_1$	$0.4\gtrsim -\delta_1^{\tau}\gtrsim\delta_1^{\alpha}>0$ $\lvert\epsilon_1\rvert<\pi/4$	$^3P_1$	$\delta_2^{\,3}\sim -0.3+\Delta_c$ $0<\Delta_c\lesssim 0.2$			$^1P_1$	$^1\delta_1=-0.2\sim -0.4$
2	$^3D_2$	$\delta_2^{\,3}=0.35\sim 0.55$	$^3P_2+{}^3F_2$	$0.15\gtrsim\delta_2^{\alpha(1)}>0^*$ $0>\delta_2^{\tau}\gtrsim -0.1$ $0>\epsilon_2$	$^1D_2$	$^1\delta_2\cong 0\sim 0.8$		
3	$^3D_3+{}^3G_3$	$\lvert\delta_3^{\alpha}\rvert,\ \lvert\delta_3^{\tau}\rvert\lesssim 0.1$ $\lvert\epsilon_3\rvert<\pi/4$	$^3F_3$	$\delta_3^{\,3}\cong -0.02$			$^1F_3$	$^1\delta_3\cong -0.075$

* $\delta_2^{\alpha}=\delta_2^{\alpha(1)}+\Delta_c$

§ 3. Proton-proton scattering

In this section, we shall analyse in detail the p p scattering data on the basis of the qualitative features of the scattering parameters shown in Table 1. Firstly the polarized cross section is analysed, because only the triplet odd state contributes to this phenomenon. The discussions are confined in the part of the nuclear scattering, i.e. in the region $\theta\gtrsim 30^\circ$, where θ is the scattering angle in the center of mass system.

A. *Proton-proton polarized cross section*

Analysis of the polarized cross section is facilitated by the fact that ϵ_2-dependence of main terms is factorized in a common function $f(\epsilon_2)$ which is defined in Appendix and plotted in Fig. 8.

The $p-p$ polarized cross section is given by

$$P(\theta)\,(d\sigma/d\Omega)_{pp}\cong (4\sin\theta/k^2)\,\{a_1^{(1)}P_1(\cos\theta)+a_3^{(1)}P_3(\cos\theta)\}, \tag{3·1}$$

where $a_1^{(1)}$ and $a_3^{(1)}$ are given in Appendix and have the form, $f(\epsilon_2) \times$ (phase shift part). The affix (1) means the total isotopic spin T of the two-nucleon system.

From the available experimental results at about 150 Mev,[11] $a_1^{(1)}$ and $a_3^{(1)}$ at 150 Mev are restricted as follows:

$$a_1^{(1)} = 0.090\,(1 \pm 0.10) \quad \text{and} \quad a_3^{(1)} = 0.014\,(1 \pm 0.2). \tag{3·2}$$

For the ${}^3P_0$- and ${}^3P_1$-states, we assume four typical sets $A^{(1)}$, $A'^{(1)}$, $B^{(1)}$ and $B'^{(1)}$ of the tensor dominant type shown in Table 2. The set $A^{(1)}$ is the most typical one of the tensor dominant type obtained by setting $\delta_0^{T(1)} = 0.50$, $\delta_1^{3(1)} = -0.30$ and $\Delta_c = 0$ in Table 1. The set $A'^{(1)}$ is the one obtained by changing Δ_c from 0 to 0.20 in $A^{(1)}$. The set $A'^{(1)}$ (the large value of δ_0^{T} and $\delta_1^{3} \sim 0$) corresponds to the circumstance that the effects of the attractive central potential (mainly due to the two-pion-exchange process) remain appreciable without being masked by some repulsive interactions in the inner region. The set $B^{(1)}$ is also one of the tensor dominant type obtained by setting $\delta_0^{T(1)} = 0.40$, $\delta_1^{3(1)} = -0.25$ and $\Delta_c = 0$. The set $B'^{(1)}$ is the one obtained by changing Δ_c from 0 to 0.2 in $B^{(1)}$. The sets $B^{(1)}$ and $B'^{(1)}$ reflect the existence of a somewhat weaker tensor force in the outer part than in the cases of $A^{(1)}$ and $A'^{(1)}$.

In each set, the scattering parameters of the ${}^3P_2 + {}^3F_2$-state (δ_2^{α}, δ_2^{T} and ϵ_2) are determined so as to be consistent

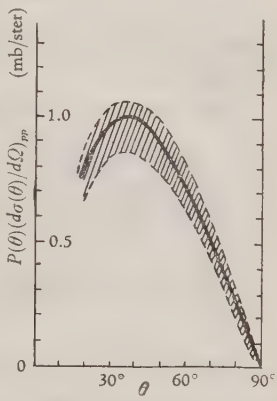

Fig. 1. $P-p$ polarized cross section at 150 Mev. The shaded area represents the available experimental data given by (3·2). The solid line is the calculated curve by use of (3·4) in the set $A^{(1)}$ shown in Table 2.

Table 2. Allowable regions of the ${}^3P_2 + {}^3F_2$-scattering parameters determined by the experimental data of the $p-p$ polarized cross section at 150 Mev (in unit of radian).

set	δ_0^{T}	δ_1^{3}	λ	*conditions given by (2·6) and $\sigma_{pp}^{\text{eff}} \cong 25$ mb		allowable regions of (δ_2^{α}, δ_2^{T}, ϵ_2)
				δ_2^{α}	$\sqrt{\sin^2\delta_2^{\alpha} + \sin^2\delta_2^{T}}$	
$A^{(1)}$	0.50	−0.30	0.38	0∼0.15	0.10∼0.20	(0.10, −0.10, 0.00) / (0.15, −0.06, −0.35)
$A'^{(1)}$	0.70	−0.08	0.54	0.20∼0.35	0.15∼0.22	(0.21, −0.08, −0.45)
$B^{(1)}$	0.40	−0.25	0.24	0∼0.13	0.20∼0.27	(No)
$B'^{(1)}$	0.60	−0.05	0.32	0.20∼0.33	0.21∼0.27	(0.20, −0.07, −0.26) / (0.27, −0.04, −0.35)

* Conditions imposed for δ_2^{T} and ϵ_2 are common for all sets, i.e. $-0.1 < \delta_2^{T} < 0$ and $\epsilon_2 < 0$.

with the experimental data. Regarding (3·2) as the equations for δ_2^{α}, δ_2^{τ} and ϵ_2, we expand the phase shift parts of $a_1^{(1)}$ and $a_3^{(1)}$ in power series of δ_2^{α} and δ_2^{τ} up to the second order, because δ_2^{α} and δ_2^{τ} are small. Then main terms in the equations become as follows;

$$\left.\begin{aligned}&\delta_2^{\alpha}\delta_2^{\tau}\simeq -(a_3^{(1)}/a_1^{(1)})\,(\lambda/5)=-0.030\,(1\pm 0.3),\\&\delta_2^{\alpha}-\delta_2^{\tau}\simeq 2a_1^{(1)}/3\lambda f(\epsilon_2)=0.060\,(1\pm 0.1)\,(1/\lambda f(\epsilon_2)),\end{aligned}\right\}\qquad(3\cdot 3)$$

where $\lambda=\sin^2\delta_0^{\tau}+(3/2)\sin^2\delta_1^{\tau}+(7/4)\sin^2\delta_3^{\tau}$ and the value of λ for each set is given in Table 2.

The conditions to determine the allowable regions of δ_2^{α}, δ_2^{τ} and ϵ_2 are as follows:

(i) $0.15\gtrsim\delta_2^{\alpha(1)}>0>\delta_2^{\tau}\gtrsim-0.1$, $0>\epsilon_2$ (Table 1).

(ii) eqs. (3·3).

(iii) The experimental value $\sigma_{pp}^{\mathrm{eff}}(1)\simeq 25$ mb[12] and the restriction $0\cdot\cdot\delta\cdot\cdot 0.4$ (see eq. (2·9)).

The allowable regions of δ_2^{α}, δ_2^{τ} and ϵ_2 are shown in Fig. 2.

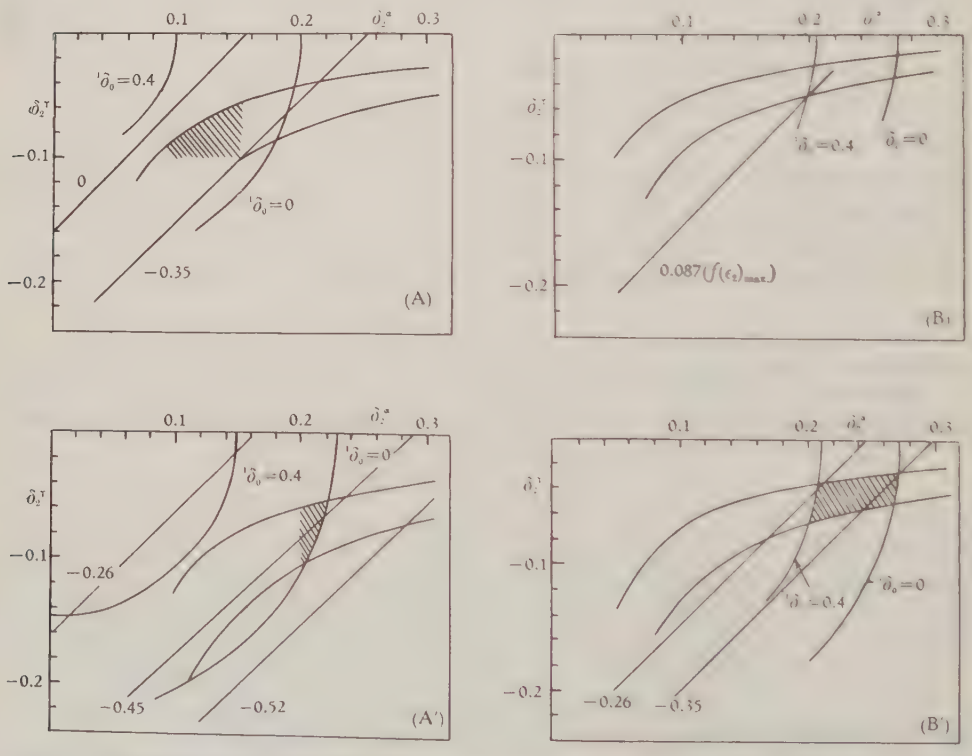

Fig. 2. Allowable regions of the $^3P_2+^3F_2$-scattering parameters determined by the $p-p$ polarization data and the $p-p$ effective total cross section. (A), (A'), (B) and (B') correspond to the sets $A^{(1)}$, $A'^{(1)}$, $B^{(1)}$ and $B'^{(1)}$ respectively. The hyperbolas and the straight lines are given by eqs. (3·3). The values attached to the straight lines are those of ϵ_2. The circles are determined by $\sigma_{pp}^{\mathrm{eff}}\simeq 25$ mb, for the two extreme cases of the $^1S_0$-phase shift, i.e. $^1\delta_0=0.4$ and 0.

For the set $A^{(1)}$, the allowable region is rather large. The best values are nearly that

$$\delta_2^{\alpha}=0.13,\quad \delta_2^{\tau}=-0.08 \quad \text{and} \quad \epsilon_2=-0.26. \tag{3·4}$$

The curve in Fig. 1 is the $p-p$ polarized cross section calculated by (3·4).

For the set $B^{(1)}$, under the condition (i), no allowable region can be found because of small λ. This fact means that, unless some interactions to make δ_2^{α} large are present as in $B'^{(1)}$, the $p-p$ polarization data cannot be reproduced by a potential with a weaker tensor force in the outer part than that of the one-pion-exchange potential with $g_e^2/4\pi=0.08$. Hereafter, we discard the set $B^{(1)}$.

B. *Proton-proton unpolarized cross section*

The $p-p$ unpolarized cross section is analysed within the allowable regions of the triplet odd state scattering parameters determined by analysing the $p-p$ polarized cross section.

The $p-p$ unpolarized cross section is expressed in the form,

$$(d\sigma(\theta)/d\Omega)_{pp}=(1/k^2)\sum_{n=\text{even}} b_n^{(1)} P_n(\cos\theta). \tag{3·5}$$

The striking feature of the experimental cross section is the isotropic angular distribution. That is, $b_2^{(1)}\sim b_4^{(1)}\sim 0$ and $b_0^{(1)}\simeq 0.73$ which is directly reduced from the effective total cross section, $\sigma_{pp}^{\text{eff}}=2\pi(d\sigma(90^\circ)/d\Omega)_{pp}\simeq 25$ mb.

We calculate $b_n^{(1)}(n=0,\ 2,\ 4)$ for the sets $A^{(1)}$, $A'^{(1)}$ and $B'^{(1)}$ given in Table 2. The ${}^1S_0$-phase shift ${}^1\delta_0$ is chosen to reproduce the experimental value of $b_0^{(1)}$. The results are shown in Table 3 and Fig. 3.

From Table 3 and Fig. 3, we find the strong ϵ_2-dependence of $(d\sigma/d\Omega)$. This comes mostly from the ϵ_2-factors of the following terms in $b_2^{(1)}$;

$$\begin{aligned}&4[\cos\epsilon_2+\sqrt{3/2}\sin\epsilon_2]^2(2\alpha|0\gamma)+4[-\sin\epsilon_2+\sqrt{3/2}\cos\epsilon_2]^2(2\gamma|0\gamma)\\&+9[\cos\epsilon_2-\sqrt{2/3}\sin\epsilon_2]^2(2\alpha|1\beta)+9[\sin\epsilon_2+\sqrt{2/3}\cos\epsilon_2]^2(2\gamma|1\beta),\end{aligned} \tag{3·6}$$

Table 3. Calculated values of $b_n^{(1)}$ (the coefficient of $P_n(\cos\theta)$ in the formula of the $p-p$ unpolarized cross section) for the sets which are compatible with the $p-p$ polarization data, at 150 Mev.

set	δ_2^{α}	δ_2^{τ}	ϵ_2	$b_0^{(1)}$	$b_2^{(1)}$	$b_4^{(1)}$	ajusted values of ${}^1\delta_0$ to $\sigma_{pp}^{\text{eff}}\simeq 25$ mb
$A^{(1)}$	0.13	−0.08	−0.52	0.75	−0.22	0.21	0.35
			−0.26		0.03	0.06	
			0.00		0.46	−0.12	
$A'^{(1)}$	0.22	−0.08	−0.52	0.74	−0.43	0.28	0.00
			−0.35		−0.25	0.09	
			−0.26		−0.12	−0.03	
$B'^{(1)}$	0.27	−0.04	−0.35	0.72	0.03	0.14	0.00
	0.21	−0.04	−0.26	0.74	0.32	0.10	0.40

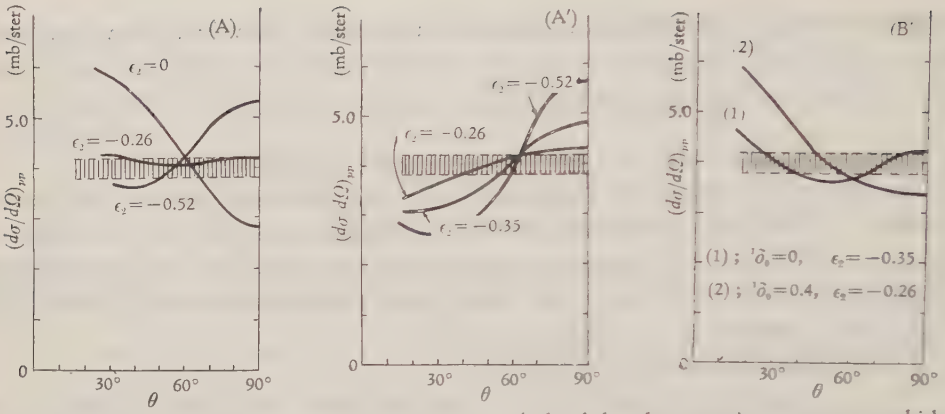

Fig. 3. $P-p$ unpolarized cross sections at 150 Mev calculated by the scattering parameters which can explain the $p-p$ polarization data and are shown in Table 2. The shaded area represents the available experimental data[12].

(A) Calculated curves for the set $A^{(1)}$.

(A') Calculated curves for the set $A'^{(1)}$.

(B') Calculated curves for the set $B'^{(1)}$. The curves (1) and (2) are calculated respectively by the upper set and the lower set in Table 3.

where $$(J_l\rho|J'\rho') \equiv \sin\delta_{J}^{\rho}\sin\delta_{J'}^{\rho'}\cos(\delta_{J}^{\rho}-\delta_{J'}^{\rho'}).$$

It is clear from (3·6) that the forward peak becomes pronounced when ϵ_2 tends to zero from negative values, because $(2\alpha|0\gamma)>0$, $(2\gamma|0\gamma)<0$, $(2\alpha|1\beta)<0$ and $(2\gamma|1\beta)>0$ in the presence of a strong tensor potential with the positive sign in the triplet odd state. Therefore, values of ϵ_2 are sharply determined by the experimental angular distribution.

Thus the $p-p$ scattering data restrict the allowable properties of the scattering parameters in the $T=1$ state as follows.

Set $A^{(1)}$: With $\epsilon_2\simeq-0.26$ the $p-p$ data are completely reproduced (Fig. 1 and Fig. 3 (A)). Roughly speaking, the singlet scattering gives the forward peak and the triplet scattering mainly due to the strong tensor force ($V_T<0$) gives a peak at $\theta\sim90°$, and as a whole the isotropic angular distribution is obtained.

Set $A'^{(1)}$: This set is excluded, because the strong peak at $\theta\simeq90°$ takes place in the restricted values of the $^1P_1$-$^1F_3$-scattering parameters which are consistent with the polarization data (Fig. 3 (A')). This fact results from the large value of the $^1P_1$-phase shift.

Set $B'^{(1)}$: One extreme case shown in Table 2 (the largest $^1\delta_0$ and the smallest δ_2^{α}) is excluded, because due to the small value of δ_1^{γ} the strong singlet forward scattering cannot be cancelled. In the other extreme case (the smallest $^1\delta_0$ and the largest δ_2^{α}), an interference minimum at $\theta\sim50°$ appears due to a small singlet scattering (Fig. 3 (B')). Therefore, the set $B'^{(1)}$ is more unfavorable compared with the set $A^{(1)}$, although this set is not completely excluded by the data.

From the results of this section, we can conclude that the sets with the same

qualitative property as A^{+} are most favorable to explain the $p-p$ data at 150 Mev. The set A^{+} is characterized by a strong tensor force just like the one-pion-exchange potential in the outer part. It should be emphasized that this strong tensor force with the positive sign is indispensable to explain the large polarization for $\theta \leq 90^{\circ}$. This fact constitutes a strong support to the pion theory of nuclear forces. Besides, two properties indicated by $\delta_0 \sim 0$ and the small $^1S_0$-phase shift are to be noted, since both facts may be the positive evidences for the existence of a hard-core-like repulsive interaction, as will be discussed in § 5.

§ 4. Neutron-proton scattering

In this section, the $n-p$ polarized and unpolarized cross sections are analysed to investigate properties of the interaction in the $T=0$ state. Analyses are performed using the set of the $T=1$ scattering parameters summarized in Table 4. Firstly, we shall analyse the $n-p$ polarized cross section in order to find gross allowable features of the scattering parameters in the triplet even state, though the available experimental data are not so accurate. Then the $n-p$ unpolarized cross section will be analysed in detail. For the sake of convenience of analyses, we neglect the $^1G_4$-wave at first and take it into consideration later on.

A. *Neutron-proton polarized cross section*

The $n-p$ polarized cross section is written in the form,

$$P(\theta)\,(d\sigma/d\Omega)_{np} = (\sin\theta/k^2)\sum_n a_n P_n(\cos\theta). \tag{4·1}$$

The expressions of a_n are given in Appendix. For $n=$odd, we divide a_n into $a_n^{(0)}$ (the part of the $T=0$ state) and $a_n^{(1)}$ (the part of the $T=1$ state). As the values of $a_n^{(1)}$, we adopt the experimental values of the $p-p$ polarization given by (3·2).

The experimental data[7] shown in Fig. 4 have the features that $P(\theta) < 0$ at $\theta \lesssim 110^{\circ}$ and $P(\theta) > 0$ at $\theta \lesssim 110^{\circ}$. Therefore, the experimental values of a_n are roughly represented as follows:

$$a_1 = 0.4 \sim 0.8, \quad 0 < (a_3, a_5) \lesssim 1/2 \tag{4·2}$$

and both a_2 and a_4 are small compared with a_1.

The following terms are important to give the large and positive a_1.

$$(\rho(\epsilon_2)/4)\{-9[1a\ 1\gamma]+5[2\beta\ 1a]-5[2\beta\ 1\gamma]\}, \tag{4·3}$$

where $\rho(\epsilon_2)$ is defined in Appendix and plotted in Fig. 8.

Under the conditions of Table 1, $\delta_0 \sim 1$, $\epsilon_1 \sim 0$ and $\epsilon_2 \sim 0$, so all terms in the bracket (4·3) contribute additively to a_1. Therefore the experimental fact, $a_1 \lesssim 0.4$, requires $0 \lesssim \epsilon_2 \lesssim 0.35$ and a large value of $\delta_2^{(1)}$.

Now choose the following sets as typical ones. They are adjusted to give the experimental $n-p$ total cross section:

$$\begin{aligned} &\text{Set } A^{(0)}\ ; \quad \delta_2{}^{\gamma}=0.55, \quad \delta_1{}^{\alpha}=0.20, \quad \delta_1{}^{\gamma}=-0.40, \\ &\text{Set } B^{(0)}\ ; \quad \delta_2{}^{\beta}=0.45, \quad \delta_1{}^{\alpha}=0.40, \quad \delta_1{}^{\gamma}=-0.40, \\ &\text{Set } C^{(0)}\ ; \quad \delta_2{}^{\gamma}=0.45, \quad \delta_1{}^{\alpha}=0.20, \quad \delta_1{}^{\gamma}=-0.60. \end{aligned} \tag{4·4}$$

For each set, $\epsilon_1=0$, 0.35 and $\delta_3{}^{\alpha}=$ 0.1, 0, -0.1 are assumed. The calculated curves of the $n-p$ polarized cross section are shown in Fig. 4.

From Fig. 4, the following results are obtained :

The tensor dominant character of the scattering parameters ; All sets $A^{(0)}$, $B^{(0)}$ and $C^{(0)}$ can reproduce the $n-p$ polarization data if $\delta_3{}^{\alpha}>0$, because $\delta_2{}^{\beta}$ and $|\delta_1{}^{\gamma}|$ are large. This property of the triplet even phase shifts comes from the strong tensor force just like the one-pion-exchange potential. Further, it should be noted that the tensor dominant character of the triplet odd scattering parameters is important to explain $a_0>0$, as is understood from the results obtained at 90 Mev by Watari.[3]

Allowable region of ϵ_1 ; The $n-p$ polarization data restrict ϵ_1 as $0\lesssim\epsilon_1\lesssim 0.35$, though they are not so sensitive to the change of ϵ_1 in this region because the ϵ_1-dependence of a_0 and a_1 is mainly attributed to $g(\epsilon_1)$.

$^3D_3+{}^3G_3$-state ; The sets with $\delta_3{}^{\alpha}<0$ are incompatible with the experimental data, because they give a positive polarization for $\theta\gtrsim 120^\circ$ due to their large a_0/a_1 ($>1/2$) and negative a_3. The fact, $\delta_3{}^{\alpha}>0$, requires $\delta_3{}^{\gamma}>0$ and $\epsilon_3>0$ under the presence of a negative and strong tensor force as the one-pion-exchange potential. The inclusion of effects of the $^3G_3$-wave determined by the pion-theoretical potential

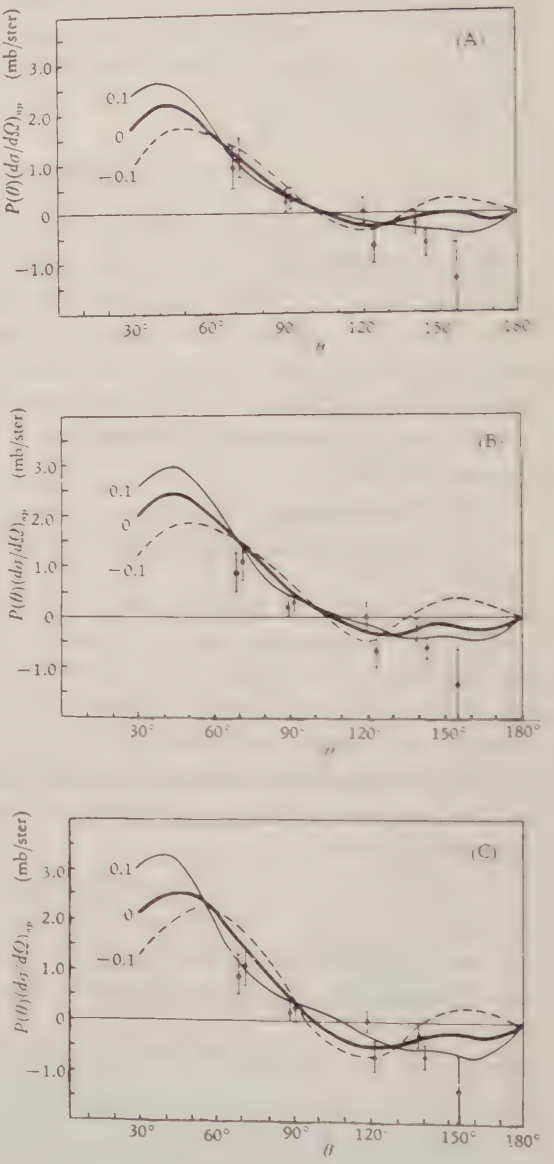

Fig. 4. Calculated curves of the $n-p$ polarized cross section at 150 Mev, for three sets given by (4·4). (A), (B) and (C) correspond to the sets $A^{(0)}$, $B^{(0)}$ and $C^{(0)}$, respectively. The experimental points are those in ref. 13). Numbers attached to the curves are values of $\delta_3{}^{\alpha}$.

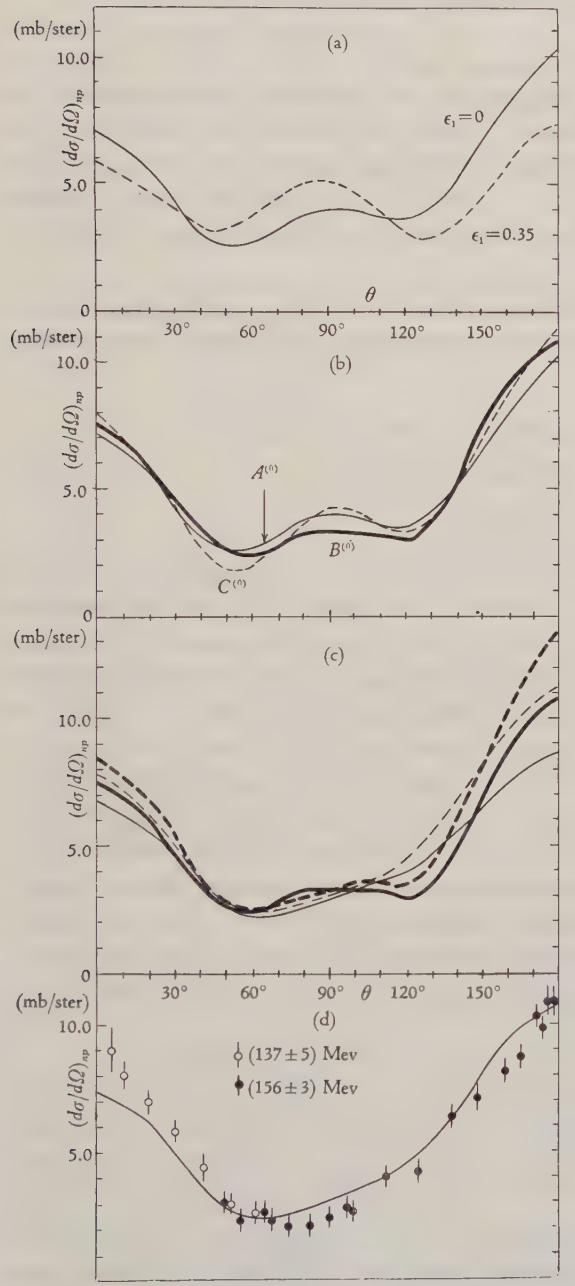

Fig. 5. $N-p$ unpolarized cross section at 150 Mev.
(a) Calculated curves by the set $A^{(0)}$ for two values of ϵ_1, 0 and 0.35.
(b) Calculated curves for three sets with $\epsilon_1=0$.
(c) Effect of the ${}^3G_3$-wave (for the set $B^{(0)}$ with $\epsilon_1=0$).

makes the $n-p$ polarization better. Effects of the ${}^3G_3$-wave are no more discussed here and will be investigated in the next part, since the $n-p$ polarization data are not accurate.

B. *Neutron-proton unpolarized cross section*

In this part, by comparison with the $n-p$ unpolarized scattering data,[14] we shall investigate the following problems ; to clarify which sets among $A^{(0)}$, $B^{(0)}$, and $C^{(0)}$ is preferable for the angular distribution, to determine ϵ_1 more precisely, to investigate in detail the effect of the ${}^3G_3$-wave and to discuss the interaction in the singlet odd state by means of the ${}^1P_1$-phase shift.

The $n-p$ unpolarized cross section is expressed as

$$(d\sigma(\theta)/d\Omega)_{np} = (1/4k^2)\sum_n b_n P_n(\cos\theta), \qquad (4\cdot5)$$

where $b_n=b_n^{(0)}+b_n^{(1)}$ for $n=$even. For $b_n^{(1)}$ we substitute the values of the set $A^{(1)}$ with $\epsilon_2=-0.26$ given in Table 3.

The experimental angular distribution is roughly symmetric about $\theta\cong 70°$, and is slightly larger in the backward direction.[14] Therefore, the following conditions,

——; ${}^3G_3$-wave not included, ${}^1\delta_1=-0.2$.
— —; ${}^3G_3$-wave ($\delta_3{}^T=-0.05$, $\epsilon_3=0.52$) included, ${}^1\delta_1=-0.2$.
- - -; ${}^3G_3$-wave not included, ${}^1\delta_1=-0.4$.
– – –; ${}^3G_3$-wave ($\delta_3{}^T=-0.05$, $\epsilon_3=0.52$) included, ${}^1\delta_1=-0.4$.
(d) Comparison with the experimental data. The calculated cross sections are given by the scattering parameters in Table 4.

$$b_2 \geqslant 0,\ b_1 < 0 \text{ and } |b_n/b_2| < 1 \quad (n=1,\ 3,\ 4) \tag{4·6}$$

should be satisfied.

Analyses are performed under the conditions given in Table 1 and on the basis of the results hitherto obtained. At first, assuming ${}^3\delta_3 = 0$ and $\epsilon_3 = 0$, we calculate b_n for the sets $A^{(0)}$, $B^{(0)}$ and $C^{(0)}$ given by (4·4). In the calculations, two values of ϵ_1 (0 and 0.35) are adopted, because the $n-p$ polarization data restrict ϵ_1 as $0 \lesssim \epsilon_1 \lesssim 0.35$. At first, the ${}^1P_1$-phase shift is fixed as ${}^1\delta_1 = -0.20$. Then the ${}^3G_3$-wave effects are taken into account. Finally, the ${}^1P_1$-phase shift is changed in the region shown in Table 1. The results are shown in Fig. 5.

The large ${}^3\delta_2$ mainly affected by a strong tensor force plays an important role to explain the peaks in the forward and backward directions. The ${}^3D_2$-state contributes mostly in the form:

to b_2,
$$(25/14)\,(2_1\beta|2_1\beta) + 10\,(\cos\epsilon_1 - 2^{-1/2}\sin\epsilon_1)^2\,(2_1\beta|1\alpha) + 10\,(\sin\epsilon_1 + 2^{-1/2}\cos\epsilon_1)^2\,(2_1\beta|1\gamma) \tag{4·7}$$

and to b_4,
$$(40/7)\,(2_1\beta|2_1\beta). \tag{4·8}$$

The large ${}^3\delta_2$, however, makes b_4 large, so $(d\sigma(\theta)/d\Omega)_{np}$ has the tendency to show a hump at $\theta \cong 90^\circ$ and dips at $\theta \cong 50^\circ$ and 130°. In order to suppress this tendency and explain $(d\sigma(90^\circ)/d\Omega) \cong 2.5$ mb ster, large values of b_2 are indispensable. To make the situation best, the triplet even scattering parameters must have the following properties:

Mixing ratio with $J=1$: $\epsilon_1 \cong 0$ is indispensable for all sets because of $(2_1\beta|1\alpha) > 0$ and $(2_1\beta|1\gamma) < 0$, as clearly understood from the dependence of (4·7) on ϵ_1 (Fig. 5 (a)).

Selection of the sets: The set $B^{(0)}$ is the best, because it satisfies the condition $(b_4/b_2) < 1$, while the other sets $A^{(0)}$ and $C^{(0)}$ do not (Fig. 5 (b)). Therefore, the properties, ${}^3\delta_2 = 0.40 \sim 0.45$ and ${}^1\delta_1{}^\alpha \sim -{}^1\delta_1{}^\gamma \sim 0.4$, are considered to be required from the $n-p$ unpolarized scattering data. It is not favorable if the two-pion-exchange central potential in the triplet even state is strongly attractive because ${}^3\delta_2$ becomes too large at high energies ($E_{\text{lab}} \gtrsim 150$ Mev). Therefore, Brueckner and Watson's[17] and Gartenhaus' potentials[19] are unreasonable from this reason in addition to the theoretical reasons discussed in ref. 18a).

In the case of $B^{(0)}$ with $\epsilon_1 = 0$, further results are obtained as follows:

${}^3G_3$-wave effects: Even in the set $B^{(0)}$, the hump at $\theta \sim 90^\circ$ and the dips at $\theta \cong 50^\circ$ and 130° do not disappear completely. This difficulty is removed by including effects of the ${}^3G_3$-wave of the pion theoretical type, that is, $\epsilon_3 > 0$ and $-0.1 \lesssim {}^3\delta_3 < 0$. For example, Fig. 5 (c) shows their effects in the case of $\epsilon_3 = 0.52$ and ${}^3\delta_3 = -0.05$.

${}^1P_1$-wave phase shift: As is seen from Fig. 5 (c), although the inclusion of the ${}^3G_3$-wave makes the fit better, it makes both the forward and backward peaks weaker than the experimental ones. Noticing that the experimental backward peak is slightly stronger than the forward one, we get the final fit by modifying the ${}^1P_1$-phase shift, that is, by changing ${}^1\delta_1$ from

$${}^1\delta_1 = -0.2 \text{ to } {}^1\delta_1 = -0.35 \text{ (Fig. 5 (d))}.$$

Thus the final set of scattering parameters given in Table 4 can completely reproduce the $n-p$ unpolarized cross section.

Summarizing the results of this section, the following conclusions are obtained. The tensor dominant character of the scattering parameters (especially, the large values of $\delta_2{}^3$ and $|\delta_1{}^\tau|$ in addition to the large separation of the ${}^3P_J$-phase shifts) is of particular importance. Also, the facts, $\delta_1{}^\alpha \sim 0.4$ and ${}^1\delta_1 \sim -0.35$, give valuable information to the nuclear interaction in the $T=0$ state, as will be discussed in the next section.

Table 4. Final values of the scattering parameters which are consistent to all experimental data at 150 Mev (in radian).

S \ J	1 (triplet)				0 (singlet)			
	T = 0		T = 1		T = 1		T = 0	
0			${}^3P_0$	$\delta_0{}^\tau \cong 0.50$	${}^1S_0$	${}^1\delta_0 \sim 0.3$		
1	${}^3S_1+{}^3D_1$	$\delta_1{}^\alpha \sim 0.4$ $\delta_1{}^\tau \sim -0.4$ $\epsilon_1 \sim 0$	${}^3P_1$	$\delta_1{}^3 \cong -0.30$			${}^1P_1$	${}^1\delta_1 \cong -0.4 \sim -0.3$
2	${}^3D_2$	$\delta_2{}^3 = 0.40$ ~ 0.45	${}^3P_2+{}^3F_2$	$\delta_2{}^\alpha \cong 0.13$ $\delta_2{}^\tau \cong -0.08$ $\epsilon_2 \cong -0.26$	${}^1D_2$	${}^1\delta_2 \cong 0.08$		
3	${}^3D_3+{}^3G_3$	$\delta_3{}^\alpha \sim 0.1$ $\delta_3{}^\tau = -0.05 \sim -0.1$ $\epsilon_3 \sim 0.5$	${}^3F_3$	$\delta_3{}^3 \cong -0.02$			${}^1F_3$	${}^1\delta_3 \cong -0.075$

§ 5. Properties of the nuclear interaction indicated by the scattering parameters

The characteristic features of the scattering parameters obtained in the preceding two sections imply various important properties of the nuclear interaction, as we discuss below.

The tensor dominant character of the scattering parameters is the most decisive factor that serves to reproduce all the important qualitative aspect of the experimental data at 150 Mev. The values of the triplet scattering parameters shown in Table 4 are very close to those which are mainly determined by the outer part of the tensor force of the one-pion-exchange potential. From these results, we can conclude that the nuclear interaction has the following properties :

(1) The tensor dominant feature of the outer part of the pion-theoretical nuclear forces is of decisive importance still at high energies.

(2) We can find no evidences for the existence of such velocity-dependent interactions as play an essential role in scatterings up to 150 Mev.

(3) A hard-core-like repulsive interaction probably exists in all states as is discussed below.

(i) In the singlet even state, the small value of the $^1S_0$-phase shift ($^1\delta_0 \sim 0.3$ at 150 Mev) indicates the existence of a hard-core-like repulsive interaction, as introduced originally by Jastrow.[17]

(ii) In the triplet even state, the rapid decrease of the $^3S_1$-phase shift at high energies ($\delta_1^{\alpha} \sim 0.4$ at 150 Mev) indicates that a hard-core-like repulsive interaction exists also in this state. This conclusion agrees with the results obtained by Watari at 90 Mev.[3]

(iii) In the triplet odd state, the fact $J_J \sim 0$ is the evidence for the existence of a hard-core-like repulsive interaction as follows: $J_J \sim 0$ means that the effect of the central force to the ^3P_J-phase shifts is roughly cancelled out as a whole. From the analyses in the low energy region, the strong attractive property of the two-pion-exchange potential has been clearly verified. Therefore, we can expect the existence of a hard-core-like repulsive interaction which suppresses the increase of the ^3P_J-phase at high energies due to this attractive force.

(iv) In the singlet odd state, the $^1P_1$-phase shift ($^1\delta_1 \sim -0.35$ at 150 Mev) seems to be affected by some repulsive interaction at small inter-nucleon distances in addition to the strong repulsive one-pion-exchange potential at large distances. Also a hard-core-like repulive interaction in this state seems to be favorable in explaining the backward peak of the $n-p$ angular distribution at high energies.

§ 6. Critique on Signell and Marshak's and Gammel and Thaler's spin-orbit coupling potentials

In the works made by Signell and Marshak[6] and by Gammel and Thaler,[5] strong spin-orbit coupling potentials are considered to be indispensable to explain the high energy data. However, the spin-orbit coupling potentials they introduced are too strong to be expected from the present pion theory at large inter-nucleon distances. In fact, a recent calculation made by S. Sato et al.[7] as well as other perturbational calculations[8] show that the spin-orbit potential is due to some retardation effects and is small compared with the static pion-theoretical potentials[18] at $x \gtrsim 0.7$. These situations are shown in Fig. 6 and in Table 5. Although there probably exist not only the spin-orbit potential but also some other velocity-dependent interactions with more complicated natures, it would be reasonable to expect that they are mostly confined at the small distances $x \lesssim 0.7$. On the other hand, the results of our work show no positive evidences for the existence of any strong velocity-dependent interactions in the outer part, that play an important role to reproduce the experimental data up to about 150 Mev.

Table 5. The strengh of spin-orbit coupling potentials in the triplet odd state (in Mev).

	pion-theoretical*	Signell-Marshak	Gammel-Thaler
$x=1$	−1.08	−8.10	−7.20
$x=2$	−0.008	−0.42	−0.019

* ref. 6) and 18a)

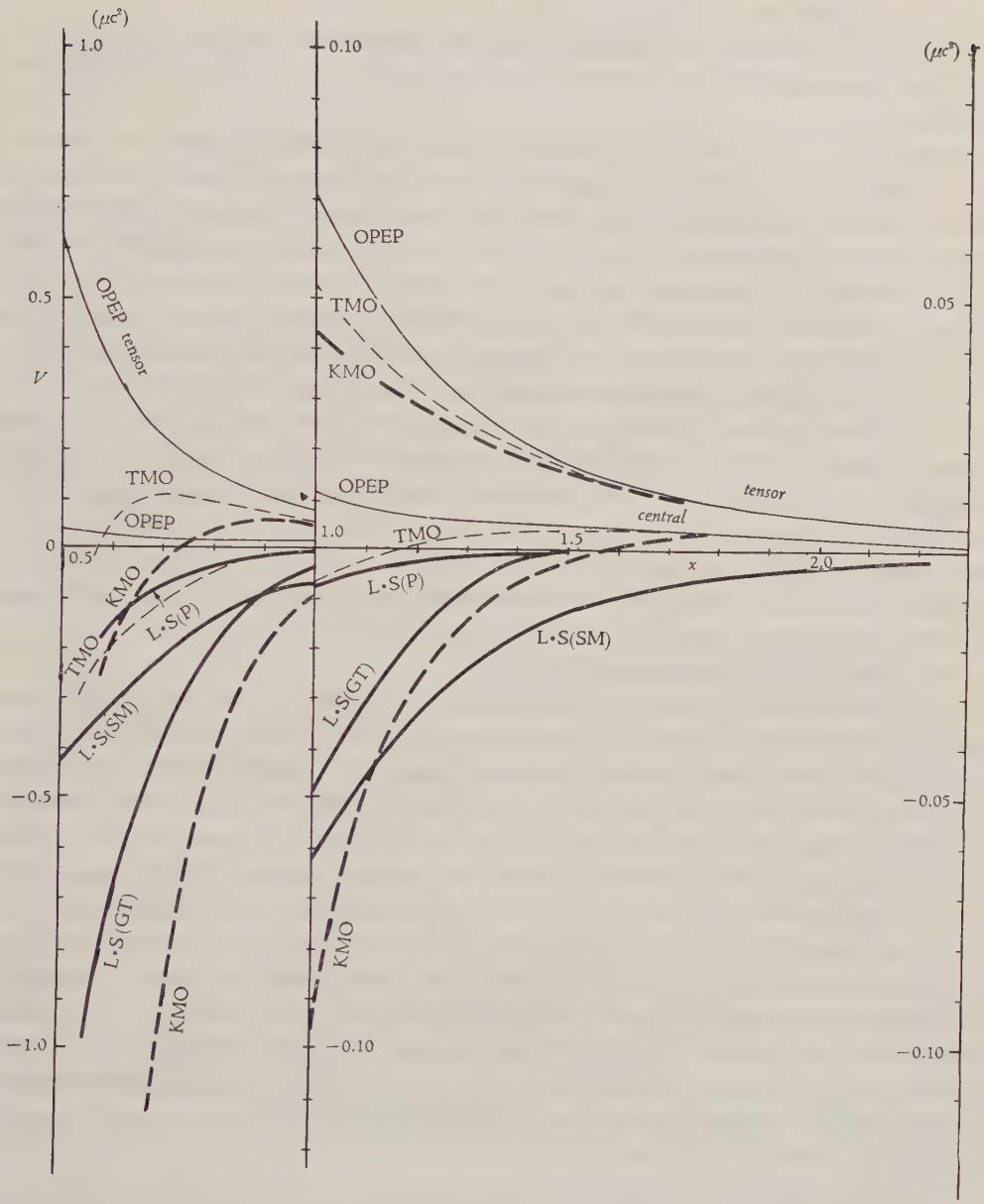

Fig. 6. Spin-orbit potentials and static pion-theoretical potentials in the triplet odd state.
L·S(P) ; the spin-orbit potential derived by the pion theory. The plotted curve is of ref. 6).
L·S(SM) ; Signell and Marshak's spin-orbit potential.
L·S(GT) ; Gammel and Thaler's spin-orbit potential.
OPEP ; the one-pion-exchange potential with $g_e^2/4\pi = 0.08$
TMO[18b], KMO[18c] ; the one- plus two-pion-exchange potentials. On the other pion-theoretical potentials, see ref. 1) and 18a).

A. *On Signell and Marshak's spin-orbit coupling potential*[4]

Signell and Marshak got a good fit with the experimental data up to 150 Mev by adding an unreasonably strong and long-range spin-orbit coupling potential to Gartenhaus' potential.[16]

Justifying to use Gartenhaus' potential, Signell and Marshak say that "his potential gives a good fit to all the low energy data. This is encouraging since there are (in essence) no free parameters in his potential: the renormarized coupling constant and the cut off energy are taken from Chew and Low's work on photo pion production and pion-nucleon scattering". However, it is easily understood that this justification is logically in contradiction with very introduction of the spin-orbit coupling potential, since, if the justification were valid, Gartenhaus' potential would be able to reproduce all experimental data.

As an immediate consequence of the above justification, the applicability of the potential, which has been pointed out and discussed in detail by Taketani[3] and other Japanese physicists,[1),18a) 20)] is not taken into consideration in their works at all.* Although Gartenhaus' potential has the correct asymptotic behaviours, their undue reliance on the inner part of the potential leads to two unaccountable facts: one is that the $^1P_1$- and $^3P_1$-states respectively have one unphysical bound level.** † The other is that, as will be explained below, a deep attractive well near the origion in the triplet odd state causes apparent disagreements[3)] with the experimental data and made it indispensable to introduce an *unreasonably strong* spin-orbit potential.

The attractive well around the origin gives a very large $^3P_2$-phase shift which is not favorable in explaining the data. It is suppressed by the spin-orbit potential. To avoid the unphysical bound states without destroying a good fit obtained by such a potential, they adopted the zero cut-off procedure. They also applied the same zero cut-off for $x \leq 0.57$ to Gartenhaus' potential, and found that the more reduction of the $^3P_2$-phase shifts is necessary. This is apparent, because the resulting values of the $^3P_2$-phase shifts are very near those of the set $A'^{(1)}$ in Sec. 3 (compare the points denoted by G' in Fig. 7 and those in Table 2). The adoption of the zero cut-off or, more generally, any prescription of the inner interactions is purely of phenomenological nature. Although they adhere to the introduction of a strong spin-orbit potential to reduce the $^3P_2$-phase shifts further, the hard-core cut-off procedure just as done in our analysis is exceedingly preferable for this purpose. If the hard-core cut-off procedure is applied to Gartenhaus' potential instead of the zero cut-off, we get the $^3P_2$-phase shifts of the tensor dominant type such as the set $A^{(1)}$ in Sec. 3.

* It is to be noted that the cut-off procedure in the momentum space adopted in Chew and Low's work is merely of phenomenological nature. Therefore, the part of the nuclear potential which seriously deyends on the cut-off procedure should be checked by comparison with the experimental data.

** In the presence of the bound levels Signell and Marshak's phase shifts of the $^1P_1$- and $^3P_2$-states are equal to π at the zero energy. In the actual calculation, they used the difference from π for the phase shifts instead of the actual ones. Therefore, the resultant numerical values would not be reliable.

† The bound level in the $^3P_2$-state is caused by an additive effect of the two deep attractive wells, one is that of Gartenhaus' potential and the other of the spin-orbit term introduced.

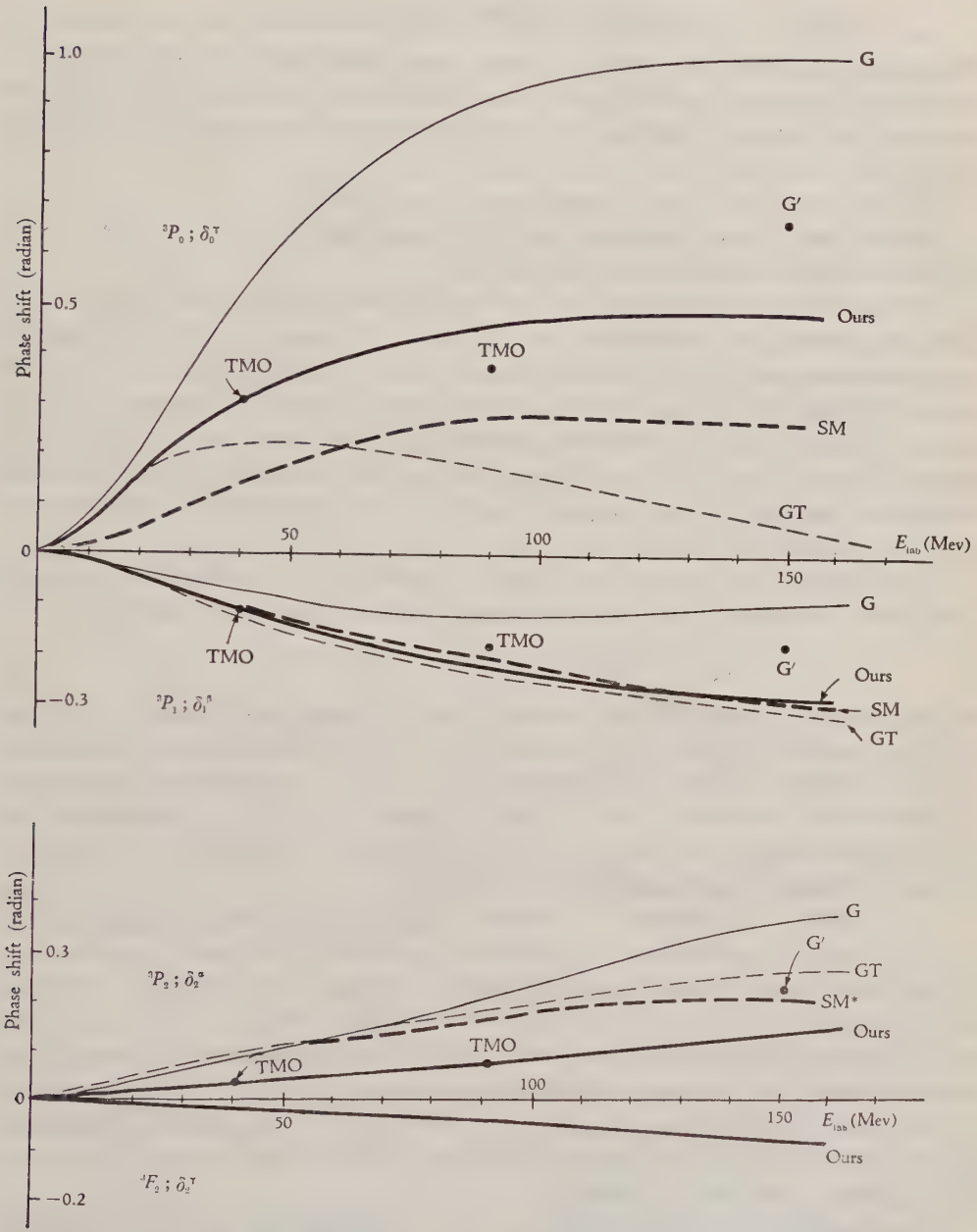

Fig. 7. ^3P_J-phase shifts calculated by various potentials, upto 150 Mev.
Ours; The values at the lower energies then 100 Mev are those obtained in ref. 1), 2) and 3). The dotted points denoted by TMO are the values calculated by the TMO potential[18b] with the hard-core cut off for $x \leq 0.333$.[21]
G; Gartenhaus'. G′; The values calculated by applying the zero cut-off to Gartenhaus' potential for $x \leq 0.57$.[4]
SM; Signell and Marshak's. *For the $^3P_2$-phase shift, differences of from π are plotted.
GT; Gammel and Thaler's.

From the above discussions, we can say that the introduction of the strong spin-orbit potential in Signell and Marshak's work comes from their undue reliance to the inner part of Gartenhaus' potential and from their adhesion to the zero cut-off procedure. Of course, good fits with the data obtained by their potential do not justify the introduction at all.

B. *On Gammel and Thaler's spin-orbit coupling potential*[5]

Gammel and Thaler's prediction of a strong spin-orbit potential is based on the phase shift analysis by Stapp et al. at 310 Mev.[9]

However, it is hardly justified that the results of the phase shift analysis are regarded as the only standard of analysis, since the results is not unique. Also it is dangerous to extrapolate any results at the high energy to the low energy regions. In fact, even below 100 Mev, the strong spin-orbit potential they introduced destroys the tensor dominant feature of the ${}^3P_J$-phase shifts established by the pion theory, as already discussed by Otsuki[2] (Fig. 7). This is due to the large potential depth of their spin-orbit potential, in spite of its small range. Therefore their unreasonably strong spin-orbit potential can be hardly accepted.

If we examine their results from a purely phenomenological view-point, their fit can be understood as follows. At 150 Mev, their ${}^3P_J$-phase shifts are of the type

$$\delta_0^{\tau} \sim 0, \quad \delta_1^{\beta} \ll 0 \text{ and } \delta_2^{\alpha} \quad 0,$$

and the value of ϵ_2 is nearly equal to ours. These features of the P_J-phase shifts make λ in (3·3) small as in the set $B^{(1)}$, and the $p-p$ polarization data can be explained only in the case of a large δ_2^{α}, as seen from Fig. 2 (B). The spin-orbit potential with the negative sign contributes so as to make δ_2^{α} large.* In the $p-p$ unpolarized cross section, the effects due to decrease of δ_1^{β} are compensated by those due to increase of δ_2^{α}, and an isotropic angular distribution is obtained.

The failure in an attempt by Gammel, Christian and Thaler[10] to reproduce the $p-p$ polarization data by means of phenomenological central and tensor potentials is attributed mostly to the values of ϵ_2 (-0.540 at 160 Mev and -0.912 at 300 Mev). For such values ϵ_2, $f(\epsilon_2)$ becomes about zero or negative (Fig. 8), and consequently small or negative polarizations appear for $\theta \lesssim 90°$: This disagreement is more or less accidental, since $f(\epsilon_2)$ is sharply dependent on ϵ_2 around its roots. Before introducing a strong spin-orbit potential as to violate the outer part of the potential, properties of the inner interaction to supress the rapid decrease of ϵ_2 should be carefully examined. This problem is now under investigation.

Summarizing the above discussions we can say that Gammel and Thaler's attempt is simply one of possible ways to understand the experimental data at the high energy. At the low energy it is hardly accepted since the tensor dominant property of the ${}^3P_J$-phase shifts is violated.

* However, since their δ_2^{τ} is slightly positive at 150 Mev, the forward shift of the peak of the $p-p$ polarization ($a_3^{(1)} > 0$) cannot be reproduced.

§ 7. Concluding Remarks

Analysing the nucleon-nucleon scattering data at 150 Mev and referring to the results of analyses below 100 Mev, we can conclude as follows. The experimental data up to 150 Mev can be understood by means of two main features of nuclear forces, a strong tensor force at large inter-nucleon distances and a hard-core-like repulsive interaction at small distances. This tensor potential has the same exchange character as the one-pion-exchange potential. In the most inner region ($x \lesssim 0.3 \sim 0.4$), a hard-core-like repulsive interaction probably exists in all states. This repulsive core is considered as something closely related to the structure of nucleon. Thus we find no positive evidences on the existence of such velocity dependent interactions as play an essential role below 150 Mev.

Now it may be said that we have plenty of knowledge on nuclear forces enough to investigate problems on nuclei on the basis of two-body interactions, since nuclear structure and low energy nuclear reaction are related with rather low energy phenomena in problems of nuclear forces. Especially, the most interesting problem is what properties of many nucleon system come from the tensor dominant character of the long rang correlations. The following potential model seems to be most reasonable as the two-body interaction to be used :

The one-pion plus two-pion-exchange potentials with $g_e^2/4\pi = 0.080$ ($x \gtrsim 0.7$)
+square wells with suitable depth ($0.7 \gtrsim x \gtrsim 0.3 \sim 0.4$)
+hard-core ($x \lesssim 0.3 \sim 0.4$).

Further investigations are required to clarify the properties of the interactions at small inter-nucleon distances. One of the phenomenological attacks on this problem is to examine the correlation between the phase shifts, the rather small values of the mixing ratios (ϵ_1 and ϵ_2) and the inner interactions.

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Appendix. Formula of the polarized cross section

The polarized cross section of nucleon-nucleon scattering is expressed in the form,

$$P(\theta)\,(d\sigma/d\Omega)_{np} = (\sin\theta/k^2)\sum_n a_n P_n(\cos\theta),$$

$$P(\theta)\,(d\sigma/d\Omega)_{pp} = (4\sin\theta/k^2)\sum_{n=\mathrm{odd}} a_n P_n(\cos\theta).$$

The coefficient a_n of the Legendre polinomial $P_n(\cos\theta)$ is represented by the phase shift δ_J^{ρ} ($\rho = \alpha$, $'\beta$ and γ) and the mixing ratio ϵ_J. α and γ specify the coupled states and $'\beta$ the uncoupled state. The states with $J \leq 3$ and $L < 3$ are taken into consideration.

a_n for even (odd) n consists of the interference terms between the different (same) parity states. We decompose a_n (n=odd) into two parts $a_n^{(0)}$ and $a_n^{(1)}$, where $a_n^{(0)}$ contains only the contributions from the $T=0$ state, i. e. the triplet even state and $a_n^{(1)}$ contains only those from the $T=1$ state, i. e. the triplet odd state.

Notations used in the expressions of a_n are as follows;

$[J\rho|J'\rho'] \equiv \sin\delta_{J}^{\rho} \sin\delta_{J'}^{\rho'} \sin(\delta_{J}^{\rho} - \delta_{J'}^{\rho'})$,

$g(\epsilon_1) \equiv \cos 2\epsilon_1 + (1/2\sqrt{2}) \sin 2\epsilon_1$,

$f(\epsilon_2) \equiv \cos 2\epsilon_2 + (1/2\sqrt{6}) \sin 2\epsilon_2$,

$g'(\epsilon_1) \equiv (7/2) \cos^2\epsilon_1 + (7/4\sqrt{2}) \sin 2\epsilon_1 + (13/4) \sin^2\epsilon_1$,

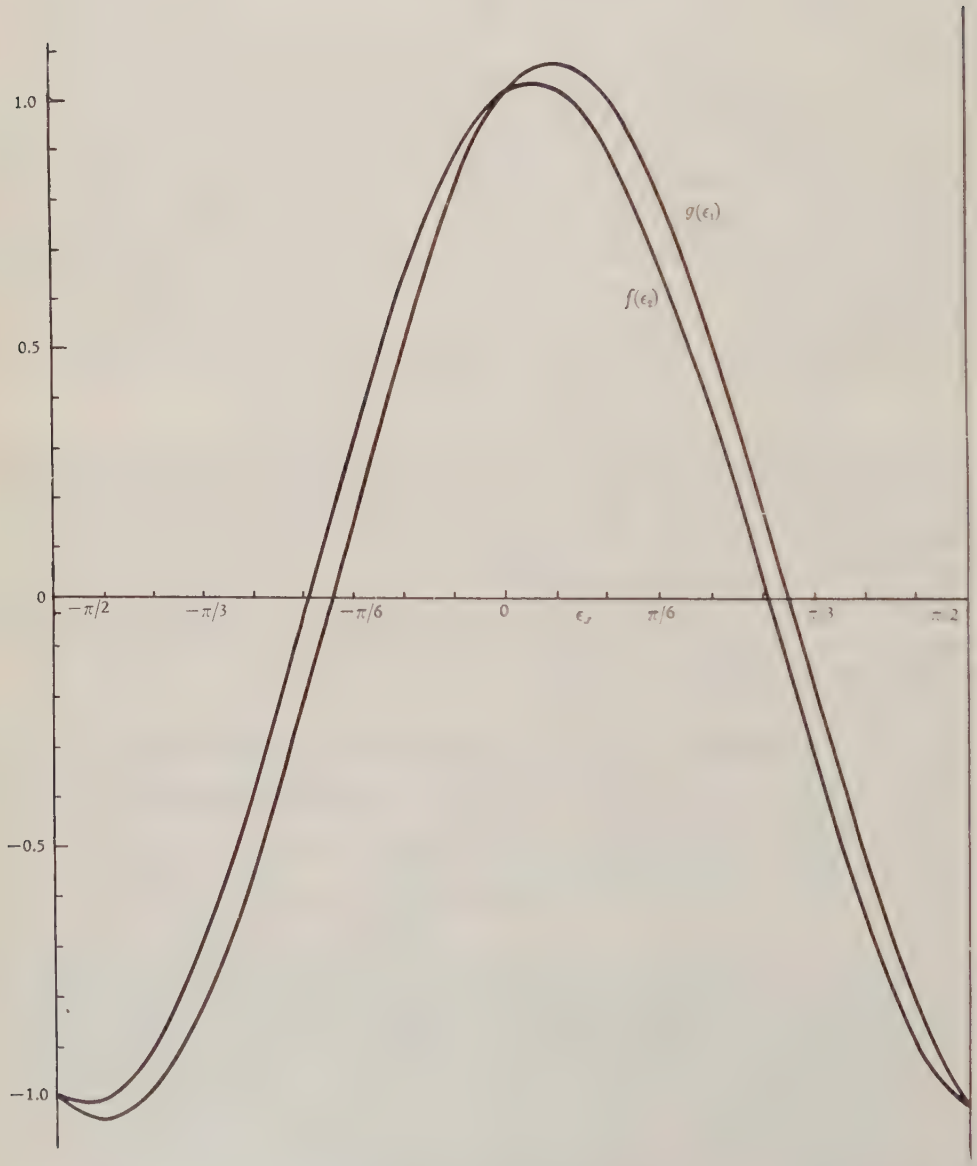

Fig. 8. Graphs of $f(\epsilon_2)$ and $g(\epsilon_1)$.

$$f'(\epsilon_2) \equiv (9/4)\cos^2\epsilon_2 + (7/4\sqrt{6})\sin 2\epsilon_2 + (13/42)\sin^2\epsilon_2,$$

$$f''(\epsilon_2) \equiv (3/4)\cos^2\epsilon_2 + (7/2\sqrt{6})\sin 2\epsilon_2 + (4/3)\sin^2\epsilon_2,$$

$$\begin{aligned} F(\epsilon_1,\ \epsilon_2) \equiv{}& (5/4)\cos^2\epsilon_1\cos^2\epsilon_2 + (5/8\sqrt{2})\sin 2\epsilon_2\cos^2\epsilon_2 + \sin^2\epsilon_1\cos^2\epsilon_2 \\ & - (5/6)\cos^2\epsilon_1\sin^2\epsilon_2 + (9/8\sqrt{6})\sin^2\epsilon_1\sin 2\epsilon_2 - (5/12\sqrt{2})\sin 2\epsilon_1\sin^2\epsilon_2 \\ & - (17/12)\sin^2\epsilon_1\sin^2\epsilon_2, \end{aligned}$$

$$\begin{aligned} G(\epsilon_1,\ \epsilon_2) \equiv{}& (9/2)\sin^2\epsilon_1\cos^2\epsilon_2 - (25/6)\cos^2\epsilon_1\sin^2\epsilon_2 + (9/4\sqrt{6})\sin^2\epsilon_1\sin 2\epsilon_2 \\ & - (25/12\sqrt{2})\sin 2\epsilon_1\sin^2\epsilon_2 - (1/3)\sin^2\epsilon_1\sin^2\epsilon_2. \end{aligned}$$

a_n $(n \leq 3)$ are represented as follows :

$$\begin{aligned} a_0 = {}& (1/2)g(\epsilon_1)\{[0\gamma|1\alpha] - [0\gamma|1\gamma]\} + (3/4)g(\epsilon_1)\{[1\beta|1\alpha] - [1\beta|1\gamma]\} \\ & + F(\epsilon_1,\ \epsilon_2)[1\alpha|2\alpha] + F(\epsilon_1,\ \epsilon_2 \pm \pi/2)[1\alpha|2\gamma] + F(\epsilon_1 \pm \pi/2,\ \epsilon_2)[1\gamma|2\alpha] \\ & + F(\epsilon_1 \pm \pi/2,\ \epsilon_2 \pm \pi/2)[1\gamma|2\gamma] + (5/4)f(\epsilon_2)\{[2\beta|2\alpha] - [2\beta|2\gamma]\} \\ & + (1/2)[0\gamma|3\alpha] + (3/4)[1\beta|3\alpha] + f'(\epsilon_2)[2\alpha|3\alpha] + f'(\epsilon_2 \pm \pi/2)[2\gamma|3\alpha] \\ & + (7/24)g(\epsilon_1)\{[3\beta|1\alpha] - [3\beta|1\gamma]\} + (4/3)[3\beta|3\alpha]. \end{aligned}$$

$$a_1 \equiv a_1^{(0)} + a_1^{(1)};$$

$$\begin{aligned} a_1^{(0)} = {}& -(9/4)g(\epsilon_1)[1\alpha|1\gamma] + (5/4)g(\epsilon_1)\{[2\beta|1\alpha] - [2\beta|1\gamma]\} \\ & + g'(\epsilon_1)[1\alpha|3\alpha] + g'(\epsilon_1 \pm \pi/2)[1\gamma|3\alpha] + (15/4)[2\beta|3\alpha], \end{aligned}$$

$$\begin{aligned} a_1^{(1)} = {}& f(\epsilon_2)\Big[(3/2)\{[0\gamma|2\alpha] - [0\gamma|2\gamma]\} + (9/4)\{[1\beta|2\alpha] - [1\beta|2\gamma]\} \\ & - (15/4)[2\alpha|2\gamma] - (21/8)\{[2\alpha|3\beta] - [2\gamma|3\beta]\} \Big]. \end{aligned}$$

$$\begin{aligned} a_2 = {}& (5/2)[0\gamma|3\alpha] + (15/4)[1\beta|3\alpha] + G(\epsilon_1,\ \epsilon_2)[1\alpha|2\alpha] + G(\epsilon_1,\ \epsilon_2 \pm \pi/2)[1\alpha|2\gamma] \\ & + G(\epsilon_1 \pm \pi/2,\ \epsilon_2)[1\gamma|2\alpha] + G(\epsilon_1 \pm \pi/2,\ \epsilon_2 \pm \pi/2)[1\gamma|2\gamma] \\ & + (5/2)f(\epsilon_2)\{[2\beta|2\alpha] - [2\beta|2\gamma]\} + f''(\epsilon_2)[2\alpha|3\alpha] + f''(\epsilon_2 \pm \pi/2)[2\gamma|3\alpha] \\ & + (35/24)g(\epsilon_1)\{[3\beta|1\alpha] - [3\beta|1\gamma]\} + (35/12)[3\beta|3\alpha]. \end{aligned}$$

$$a_3 \equiv a_3^{(0)} + a_3^{(1)};$$

$$a_3^{(0)} = (27/4)\{\sin^2\epsilon_1[1\alpha|3\alpha] + \cos^2\epsilon_1[1\gamma|3\alpha]\} + (15/4)[3\beta|3\alpha],$$

$$a_3^{(1)} = -f(\epsilon_2)\Big[(15/2)[2\alpha|2\gamma] + (21/8)\{[2\alpha|3\beta] - [2\gamma|3\beta]\} \Big].$$

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On the Wave Propagation in the Non-Linear Fields, II

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The general scalar non-linear fields satisfying the Lorentz-covariance are classified into two groups, the semi-linear and the quasi-linear. The singularities on wave fronts are investigated in each case, on the basis of the characteristic theory of partial differential equations.

§ 1. Introduction

In a previous paper[1]* of the present author, the propagation of waves in the two typical non-linear fields was investigated, one of which is of the so-called Born type and the other corresponds to the relativistic hydrodynamics, and it was shown that there exists a remarkable difference between the propagation characters of these two non-linear fields. In the present article, we shall deal with the general non-linear scalar equations and investigate the singularities on the wave front invading the vacuum, hoping that our results may be of use to attempts of explaining the multiple production of mesons from the viewpoint of non-linear theory. In our opinion, the conclusion of this paper may also be instructive for the non-linear field theory of elementary particles.

In the next section, we shall classify the non-linear equations into two groups, the semi-linear and the quasi-linear, other equations which do not belong to these groups seem to be unphysical and may be excluded from our discussion.

In § 3, the semi-linear equations will be studied not only in one dimensional but also in three dimensional cases, and it will be shown that in the both cases, the initial discontinuities on the wave front are preserved in the course of propagation. As a result the propagation character of the semi-linear equation is not so different from that of the linear equation, so far as the behaviours of the singularity on the wave front are concerned.

In § 4 we shall investigate the one dimensional propagation characters of the quasi-linear equations, to the special examples of which correspond the fields considered in the previous article (I).*

§ 2. The classification of the non-linear equations

In this paper we consider the field equations derived from the Lagrangians $L(\phi, \phi_\mu)$

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* Hereafter, this paper will be denoted as (I).

in which ϕ is a scalar field variable and ϕ_μ is its derivative with respect to the space time variable x_μ.* Hereafter, Greek subscripts assume values ranging from 1 to 4, and a repeated indces indicate summation.

Without loss of generality it may be assumed that the invariant L which gives the second order covariant equation for ϕ, is to be a function of ϕ and $Q=\phi_\mu \phi_\mu$ 2, i.e $L=L(Q, \phi)$.

The Euler equation, then, can be written as follows

$$\frac{\partial L}{\partial Q}\phi_{\mu\mu}+\frac{\partial^2 L}{\partial Q^2}\phi_\mu \phi_\nu \phi_{\mu\nu}+2Q\frac{\partial^2 L}{\partial Q \partial \phi}-\frac{\partial L}{\partial \phi}=0, \tag{2·1}$$

or

$$g_{\mu\nu}\phi_{\mu\nu}+f(\phi_\mu, \phi)=0, \tag{2·2}$$

in which $g_{\mu\nu}$ and f are given by

$$g_{\mu\nu}=\frac{\partial L}{\partial Q}\delta_{\mu\nu}+\frac{\partial^2 L}{\partial Q^2}\phi_\mu \phi_\nu,$$

and

$$f=2Q\frac{\partial^2 L}{\partial Q \partial \phi}-\frac{\partial L}{\partial \phi}$$

respectively.

According to the usual mathematical terminology, we call the equation of the type (2.2) "the quasi-linear equation", whose characteristics determined by $g_{\mu\nu}$ depend, in general, on the field variables ϕ and ϕ_μ and imply that in such a field the signal velocity may exceed the light velocity.[2] Especially if L is a linear function of Q, $g_{\mu\nu}$'s are independent of ϕ and ϕ_μ and (2.2) reduces to

$$\Box\phi+h(\phi, \phi_\mu)=0.$$

We call the equation of the above type "the semi-linear equation" whose characteristics are given *a priori* in the four dimensional coordinate space (Minkowski's light cone.) As is obvious from the above definition, the semi-linear equation should be regarded as the special case of the quasi-linear equation. Throughout this paper, however, only those equations whose characteristics are dependent on the field variables will be called "quasi-linear" and distinguished from the semi-linear equations.

Since in many cases the characteristic manifolds can be the separation surfaces, in passing over which the higher order derivatives become discontinuous, this classification of field equations seems to be most advantageous if it is applied to the investigations of the singularities on the wave fronts. Moreover the discussions along this line of classi-

* The coordinate vector of a four dimensional point x is denoted by $x_\mu(\boldsymbol{r}, it)$. The real time coordinate $x_0=(1/i)x_4=t$ is also used and we use the same unit as was used in (I) ($\hbar=c=1$).

fication may be interesting from the philosophical point of view as it is closely connected with the validity of the micro-causality.[2)]

§ 3. The semi-linear equations

In this case the Lagrangian $L(Q, \phi)$ is linear in Q, that is,

$$L=A(\phi)Q+B(\phi), \tag{3·1}$$

in which A and B are functions of ϕ alone.

In the following calculation we first consider the case in which $A(\phi)$ is equal to unity, and B is a polynomial of ϕ. The field equation, then, reduces to

$$\Box\phi-f(\phi)=0, \quad (f(0)=0) \tag{3·2}$$

in which $f(\phi)$ is equal to $B'(\phi)$. One sees that for physical applications the equation (3.2) is sufficiently general.

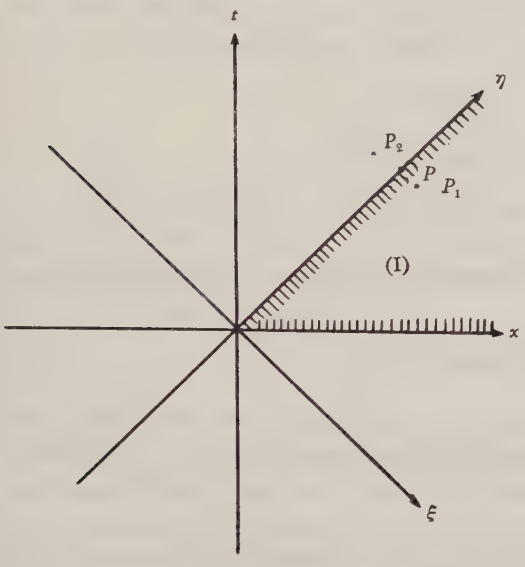

Fig. 1. The hatched region (I) is the vacuum.

In the first place, let us consider the one dimensional propagation and illustrate the behaviours of waves in the neighbourhood of the wave front facing the vacuum.

Introducing the characteristic coordinates ξ and η defined by $\xi=\frac{1}{2}(x-t)$ and $\eta=\frac{1}{2}(x+t)$ respectively, we can transform (3.2) into

$$\frac{\partial^2\phi}{\partial\xi\partial\eta}-f(\phi)=0. \tag{3·3}$$

If at $t=0$ the initial condition is given in such a way that the values of $\phi(x)$ and $\phi_t(x)$ are equal to zero for $x>0$, and finite for $x<0$, it is obvious that the region I bounded by the line $\xi=0$ and the positive part of the x-axis (c.f. Fig. I) is the vacuum and across the separation line $\xi=0$ the discontinuities generally appear in the higher order derivatives of ϕ.[3)]

However, in physical applications, one often encounters the discontinuities of ϕ_ξ or ϕ itself. Hence we begin with the analysis of the discontinuity of ϕ_ξ at $\xi=0$.

Suppose that across $\xi=0$ the function ϕ and its tangential derivative ϕ_η are continuous, while the normal derivative ϕ_ξ is discontinuous and has a jump $\Delta\phi_\xi$.

Consider equation (3.3) at points P_1 and P_2, one on each side of the line $\xi=0$. (c.f. Fig. 1) Subtract one of these equations from the other and then let P_1 and P_2 approach one point P on the line $\xi=0$.

Since ϕ and its derivatives with respect to η are continuous, we have the following equation for $\Delta\phi_\xi$:

$$d(\Delta\phi_\xi)/d\eta=0. \tag{3·4}$$

From (3.4) there results that $\Delta\phi_\xi$ is constant, equal to its initial value κ_0 and the discontinuity of the derivative ϕ_ξ is preserved in the course of the propagation.

In the limit of the infinite κ_0, the discontinuity of ϕ itself may be realized at the origin, $t=0$ and $x=0$. From the above results we, in this limit, see that the initial discontinuity of ϕ defined in this manner is preserved and propagated along the characteristic line $\hat{\xi}=0$.

It also seems to be worthwhile to observe the behaviours of the higher order derivatives, if they exist in the region of negative $\hat{\xi}$. Let us again assume that the region I is the vacuum. Differentiating once (3.3) with respect to $\hat{\xi}$ and using the same procedure as applied to ϕ_ξ, for the discontinuity of $\phi_{\xi\xi}(\Delta\phi_{\xi\xi})$, we obtain

$$\partial(\Delta\phi_{\xi\xi})/\partial\eta-\kappa_0 f'(0)=0.$$

Therefore if $f'(0)=m^2(=\text{constant})$, we have

$$\Delta\phi_{\xi\xi}=\kappa_0 m^2\eta+\kappa_1,$$

in which κ_1 is equal to the value of $\Delta\phi_{\xi\xi}$ at the origin. Hence the value of $\Delta\phi_{\xi\xi}$ is just equal to that of the linear equation,

$$(\Box-m^2)\phi=0. \tag{3·2}'$$

One can easily see that if $f(\phi)$ is the polynomial of the k-th order, the higher derivatives with respect to $\hat{\xi}$ coincide with those of the linear equation (3.2)′ up to the k-th order, and the solution in the neighborhood of the wave front can be well approximated by that of (3.2)′.

It should be noted that this result concerning the higher order discontinuities is valid on the wave front being in contact with the vacuum, if there exists a jump of the first order derivative, $\Delta\phi_\xi$. However, if all the derivatives up to the n-th order ($n\neq 0$) are continuous across the characteristic line, the jump of the $(n+1)$-th order derivative is equal to that of the linear equation (3.2)′. This conclusion is equally valid for an arbitrary state in (I), if it satisfies the above restriction.

As for the more general equation in which f is an arbitrary function of ϕ and Q, one can obtain the same result provided that the condition $f_{\substack{\phi=0\\Q=0}}=0$ is satisfied and that the singularities of ϕ_ξ on the wave front facing the vacuum are considered, because Q is equal to $\phi_\xi\phi_\eta/2$ and vanishes at $\hat{\xi}=0$ due to the continuity of ϕ_η*.

The generalization of the above result to the three dimensional propagation is easy.[4]

Let the surface $\hat{\xi}=0$ be the characteristic manifold which satisfies the differential equation

* In this paper, the existence of the solution satisfying the necessary analyticities is assumed always in the regions under consideration, though it is not obvious.

$$\tilde{\xi}_\mu \, \tilde{\xi}_\mu = 0^\dagger. \tag{3·5}$$

Construct the curvilinear coordinates η_i[††]'s and $\tilde{\xi}$ which are orthogonal to each other.

In terms of these variables, the equation (3.2) can be expressed as follows.

$$2\tilde{\xi}_\mu \, \eta_{i\mu} \, \phi_{\xi\eta_i} + \tilde{\xi}_{\mu\mu} \, \phi_\xi + \eta_{i\mu\mu} \, \phi_{\eta_i} + \eta_{i\mu} \, \eta_{k\mu} \, \phi_{\eta_i\eta_k} + f(\phi) = 0. \tag{3·6}$$

Introducing the transverse differentiation $\partial/\partial s^*$, which is the differentiation along a characteristic ray in the characteristic manifold $\tilde{\xi}=0$ and is given by $\partial/\partial s = \tilde{\xi}_\mu \, \partial/\partial x_\mu$, we have, in place of (3.6),

$$2\phi_{\xi s} + \alpha\phi_\xi + \beta = 0, \tag{3·7}$$

in which α and β are given by

$$\alpha = \tilde{\xi}_{\mu\mu}, \tag{3·8}$$

$$\beta = \eta_{i\mu\mu} \, \phi_{\eta_i} + \eta_{i\mu} \, \eta_{k\mu} \, \phi_{\eta_i} \, \phi_{\eta_k} + f(\phi). \tag{3·9}$$

Let us assume that across $\tilde{\xi}=0$ both the function ϕ and its tangential derivatives are continuous, while the normal derivative ϕ_ξ has a discontinuity $\Delta\phi_\xi$. Applying the method similar to that used in the one dimensional case and considering the fact that β varies continuously across $\tilde{\xi}=0$, we obtain

$$2 \, \partial\Delta\phi_\xi/\partial s + \alpha\Delta\phi_\xi = 0, \tag{3·10}$$

or

$$\Delta\phi_\xi = \text{const.} \cdot \exp\left(-\tfrac{1}{2}\int \alpha \, ds\right). \tag{3·11}$$

Therefore the discontinuity of ϕ_ξ is propagated according to the above equation which implies that it can never disappear. Especially, if at the initial instance $t=0$, the disturbance is localized within the sphere of the radius r_0, the equation of the wave front is given by $\tilde{\xi} = r - r_0 - t = 0$. In this case one can easily see that $\Delta\phi_\xi$ is proportional to $1/r$.

Following the same reasoning as was done in the one dimensional case, one can also say that the initial discontinuity of ϕ itself is transferred along the characteristics and can never disappear, if it is realized by making ϕ_ξ infinite.

Finally we can conclude that as for the discontinuities across the wave fronts, the non-linear equation under consideration has the same character as the linear equation.

Now, let us proceed to the discussion on the general equation derived from the Lagrangian (3.1);

$$\Box\phi + 2QA'(\phi)/A(\phi) - B'(\phi)/A(\phi) = 0.$$

In terms of the coordinates η_i's and $\tilde{\xi}$, Q can be expressed by

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† Greek subscripts denote the differentiations with respect to the space time coordinates x's.

†† Latin subscripts assume values ranging from 1 to 3 and the dummy index is also used.

* The parameter s characterizes the characteristic ray.

$$Q=\xi_\mu\,\gamma_{i\mu}\,\phi_{\eta_i}\,\phi_\xi+\gamma_{i\mu}\,\gamma_{k\mu}\,\phi_{\eta_i}\,\phi_{\eta_k}$$
$$=\phi_s\,\phi_\xi+\gamma_{i\mu}\,\gamma_{k\mu}\,\phi_{\eta_i}\,\phi_{\eta_k}.$$

Hence the equation (3.7) is also valid in this case if the expressions for α and β are to be modified as follows;

$$\alpha=\xi_{\mu\mu}+2\phi_s\,A'(\phi)/A(\phi)$$
$$\beta=\gamma_{i\mu\mu}\,\phi_{\eta_i}+\gamma_{i\mu}\,\gamma_{k\mu}\,\phi_{\eta_i}\,\phi_{\eta_k}(1+2A'(\phi)/A(\phi)).$$

Therefore the discontinuity $\Delta\phi_\xi$ can be given by (3.11), provided that A'/A and B'/A are finite on the wave front.

Especially on the wave fronts facing the vacuum 0, may, in general, be put equal to zero, and the results are the same as those previously obtained in (3.2).

In conclusion, it can be said that in the scalar semi-linear equations derived from the Lagrangian (3.1) the discontinuities across the wave fronts do not vanish in the course of propagation provided that both $A'(0)/A(0)$ and $B'(0)/A(0)$ are finite.

§ 4. The quasi-linear equations

The discussion concerning the general quasi-linear equation is very difficult because their characteristics are not given *a priori* in the coordinate space but depend on the field variables.

Hence, in this paper, we restrict our discussions to the one dimensional propagation and furthermore to the simplified case in which Lagrangian L depends only on Q and does not contain ϕ itself.

Under the assumptions introduced above, we have the following field equation,

$$(L'+L''\,\phi_x^2)\phi_{xx}-2L''\,\phi_x\,\phi_t\,\phi_{xt}-(L'-L''\,\phi_t^2)\phi_{tt}=0;$$

or in terms of u and v defined by

$$u=\phi_x \text{ and } v=-\phi_t,$$

we have

$$\begin{cases}(L'+L''u^2)\,u_x+L''uv(u_t-v_x)+(L'-L''v^2)\,v_t=0, & (4\cdot1)\\ u_t+v_x=0, & (4\cdot2)\end{cases}$$

in which L' and L'' denote respectively the first and second derivatives of L with respect to Q.

From (4.1) and (4.2) we have the equation for the characteristic direction $dx\,dt$:

$$-(L'-L''v^2)\,(dx/dt)^2-2L''uv(dx/dt)+L'+L''u^2=0, \qquad (4\cdot3)$$

or

$$dx/dt=[uv\pm\sqrt{(L'/L'')^2+(L'/L'')(u^2-v^2)}\,]/[-(L'/L'')+v^2]. \qquad (4\cdot4)$$

Corresponding to these characteristics (C-characteristics) in the coordinate space, the characteristics (Γ-characteristics) in the (u, v)-plane can be determined by

$$-(L'/L'')((du)^2-(dv)^2)=(udu-vdv)^2, \qquad (4\cdot5)$$

or

$$dv/du=[uv\pm\{(L'/L'')^2-(L'/L'')(v^2-u^2)\}^{1/2}]/[v^2-(L'/L'')]. \qquad (4\cdot6)$$

The equation (4.5) can be easily integrated in the following way. Consider, for example, the region characterized by the condition, $v>u$, then putting

$$v=q\ \cosh\theta,$$

$$u=q\ \sinh\theta,$$

one can obtain

$$(d\theta)^2=\{(1/q^2)-(L''/L')\}(dq)^2, \qquad (4\cdot7)$$

in which $Q=\frac{1}{2}q^2$, and hence L' and L'' are functions of q alone.
From (4.7) we finally have the solution

$$\theta=\pm\int\{-(L''/L')+(1/q^2)\}^{1/2}\,dq, \qquad (4\cdot8)$$

though the actual construction of solutions in this way is, in general, tedious.

However, in the case of the propagation of the waves invading the vacuum the analysis can be simplified and we can draw some consequences without specifying the actual functional form of L. In this case, one may start with the following initial conditions at $t=0$;

$$u(x,\ 0)=v(x,\ 0)=0 \quad \text{for} \quad |x|>a,$$

$$\left.\begin{aligned}u(x,\ 0)&=u_0(x)\\ v(x,\ 0)&=v_0(x)\end{aligned}\right\} \quad \text{for} \quad |x|\leqq a,$$

in which $u_0(x)$ and $v_0(x)$ are given in such a way that $\phi(x, 0)$ and $v(x, 0)$ are even functions of x and $u(x, 0)$ and $v(x, 0)$ are continuously differentiable with respect to every x. Suppose that $L'(0)\neq 0$ and denote the point $(a, 0)$ in the (x, t)-space as A.

From (4.4) we then have the two branches of the C-characteristics in the neighborhood of the point A at which there holds $u=v=0$. Hence, it may be assumed that in a sufficiently small region D in which A is included, (*c.f.* Fig. 2) there exists a unique solution continuously extended from the initial values on AB, if we assume the appropriate forms of $u_0(x)$ and $v_0(x)$ for a given Lagrangian. Of course it may be

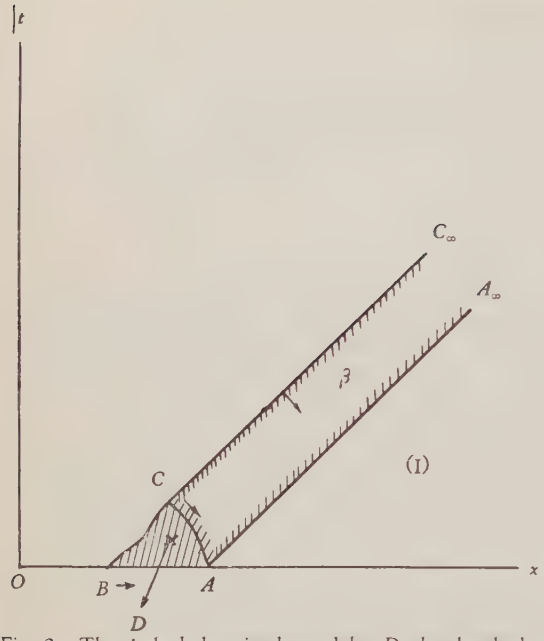

Fig. 2. The dashed domain denoted by D, the hatched region is the simple wave zone, β. The region (I) is the vacuum.

considered that for some pathological Lagrangian there does not exist any solution in the neighbourhood of A. However, it is quite obvious that there is a large number of L to which the above assumption can rigorously be applicable.

It can be seen obviously that the region I in Fig. 2 is a constant state or the vacuum*, which, by means of the formula (4.6) and the condition $L'(0) \neq 0$, is not a singular point in the (u, v)-plane; therefore the region β in Fig. 1 bounded by the two C characteristics AA_∞, CC_∞, and the C^-, AC issuing out of A, is of the simple wave type, which is to be connected continuously with the region I. Then, throughout this region, β, we have the relation $u=v$ which leads to the following expressions of the C-characteristics:

$$dx/dt \begin{cases} =1 \\ =[L'(0)+v^2L''(0)]/[-L'(0)+v^2L''(0)]. \end{cases}$$

The former equation shows that all C-characteristics are parallel straight lines, making the angle $\pi/4$ with the positive x direction. Since along each straight C both u and v are constant, the values of u and v on the line CC_∞ are equal to those at the point C. Hence, in the limit $B \to A$ (or $CC_\infty \to AA_\infty$), the discontinuities of u and v occur on the wave front AA_∞ which can never vanish in the course of the propagation, preserving their initial strength at A.

Since the Lagrangian is given, we can always define the energy density H which in some case, is a preferable quantity to the quantities u and v from the physical point of view. From the Lagrangian (1.1) we have the following expression of H;

$$H=-v^2L'-L.$$

If $L'(0) \neq 0$, in the regions where the relation, $u=v$ or $u=-v$, holds, H does never vanish as long as v is not equal to zero. (We can always adjust L in such a way that

* The relation $u=v=0$ means that $\phi=$ const., hence, it does not necessarily mean that this state is the vacuum. However, since ϕ itself does not appear explicitly in the field equation, the above constant is meaningless and can be put equal to zero without any contradiction to the field equation. Thus the state $u=v=0$ may be regarded as the vacuum.

$L(0)$ is equal to zero.) Therefore, in the limit, $B \to A$, considered above, the discontinuity of the energy density on the wave front facing the vacuum is also preserved during the propagation and characterized by the discontinuity of v.

The solution in the large can also be settled down in the following way. Let us assume that the region α can be well-defined and is bounded by the x-axis and the C^- and C^+ characteristics A_-P, A_+P, issuing out of the two points $A_+, (a, 0)$, and A_-, $(-a, 0)$, respectively. If, moreover, the unique solution extended continuously from

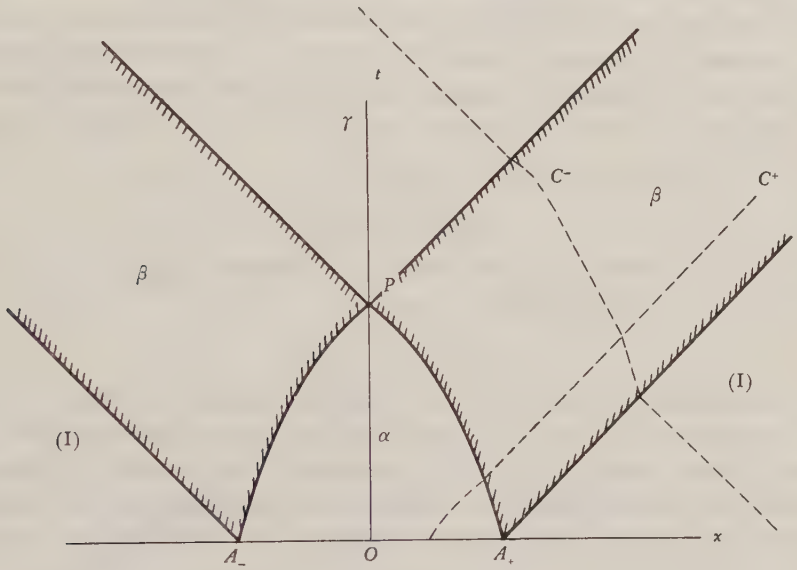

Fig. 3. The characteristics and the domains of α, β and γ.

the initial data prescribed on the portion between A_+ and A_- exists in this region, the simple wave regions ${}_i\beta^{\pm}$, adjacent to the α region and the vacuum, can also be well-defined; accordingly, as shown in Fig. 3, in the γ-region covered by the characteristic families crossing over the ${}_i\beta^{\pm}$ zones and connected with the vacuum, we obtain the relations $u=v$ and $u=-v$ or the solution $u=v=0$.

As can be seen from the above proof, the existence of the solution in the large is owing to the existence in the α-region which can be judged by using the formula (4.8) if the actual functional form of L is given. It should be emphasized that after the finite time interval the energy is contained only in the simple wave zones (the progressive zones) facing the vacuum and vanishes in the γ-region, in other words, the change in the spatial distribution of the energy can take place only at the initial stage of the propagation characterized by OP in Fig. 3 and after that any spreading of the energy does not occur and in this final process the essential character of the solution is not different from that of the linear equation derived from the Lagrangian $L=Q$.

However, if $[L'(Q)]_{Q=0}$ is equal to zero, as can be seen from (4.4) the vacuum is a singular region in which dx/dt is indefinite, hence, we cannot use the above method of solution. In such a case, in fact, we have an example[1] where the initial discontinui-

ty in the energy density is resolved immediately, and under the discontinuous initial condition, we have the continuous distribution of the energy density on the wave front facing the vacuum and have the appreciable change in the energy distribution throughout the propagation. Finally it should be noted that a Lagrangian, which is dependent on a parameter ε and tends to the linear one, Q, as ε approaches zero (or infinity), satisfies in general, the condition $L'(0) \neq 0$ and that we cannot expect for such a field to show the peculiar features characteristic in the non-linear field.

As for the non-linear Lagrangian dependent not only on Q but also on ϕ, we cannot conclude anything. However, it seems to be difficult to expect that the generalization in this way acts to resolve the singularities on the wave front facing the vacuum, if $L'\big]_{\substack{Q=0 \\ \phi=0}} \neq 0$, and if the vacuum is not the singular point of the differential equation so modified. For example, suppose that L is given by

$$L = L_1(Q) + L_2(\phi),$$

in which L_1 is a function of Q alone, and L_2 is a polynomial of ϕ.

From (2.1), the field equation becomes

$$g_{\mu\nu}\,\phi_{\mu\nu} - \partial L_2/\partial\phi = 0,$$

where $g_{\mu\nu}$ is independent of L_2. On the wave front facing the vacuum, the equation may be approximated by the first term, because in its neighbourhood one can expect $\phi = 0$ so far as the continuous solutions are considered. (Of course, we cannot exclude the possibility of the occurrence of the shock waves, the discontinuity of ϕ itself, but in this case the circumstance concerning the singularity on the wave front becomes much worse.)

§ 5. An example

Let us consider the Lagrangian

$$L = \varepsilon Q + (1/2)\,Q^2. \tag{5·1}$$

As was suggested by H. Fukuda,[5] it seems to be interesting to investigate the solution of (5.1) in the limit $\varepsilon \to 0$ because in this limit we have the condition $L'(0) = 0$ and one can expect a solution whose character is essentially different from that of (5.1) with a nonvanishing ε. Suppose that ε is negative and that the initial condition is given as follows;

at $t = 0$,

$$u = 0,$$

$$v(x) = \begin{cases} 0 & \text{for } |x| \geqq a+b \\ v_0(x) > 0 & \text{for } a < |x| < a+b \\ \alpha & \text{for } |x| \leqq a, \end{cases} \tag{5·2}$$

in which $v_0(x)$ is given in such a way that $v(x)$ is even and differentiable everywhere

with respect to x, and α is a positive constant.

Introducing the variables X and Y defined by

$$X=v-u,\quad Y=v+u,$$

and using the formula (4.8), we obtain the integrals (c.f. § 4 (I))

$$CFX=1 \quad along \quad C^-: \quad dt-\lambda_- dx=0, \tag{5·3a}$$

$$C'FY=1 \quad along \quad C^+: \quad dt-\lambda_+ dx=0, \tag{5·3b}$$

where $\lambda_\pm$ and F are given by

$$\lambda_\pm=L''uv\mp\{L'^2-L''L'(v^2-u^2)^{1/2}\}/(L''u^2+L'), \tag{5·4}$$

and

$$F=[(3q^4+8|\mathcal{E}|q^2+4\mathcal{E}^2)^{1/2}+\sqrt{3}\,q^2+(4/\sqrt{3})|\mathcal{E}|]^{\sqrt{3}/2}$$
$$\cdot[(3q^4+8|\mathcal{E}|q^2+4\mathcal{E}^2)^{1/2}+2q^2+2|\mathcal{E}|]^{-1/2}, \tag{5·5}$$

with constants of integration C and C'.

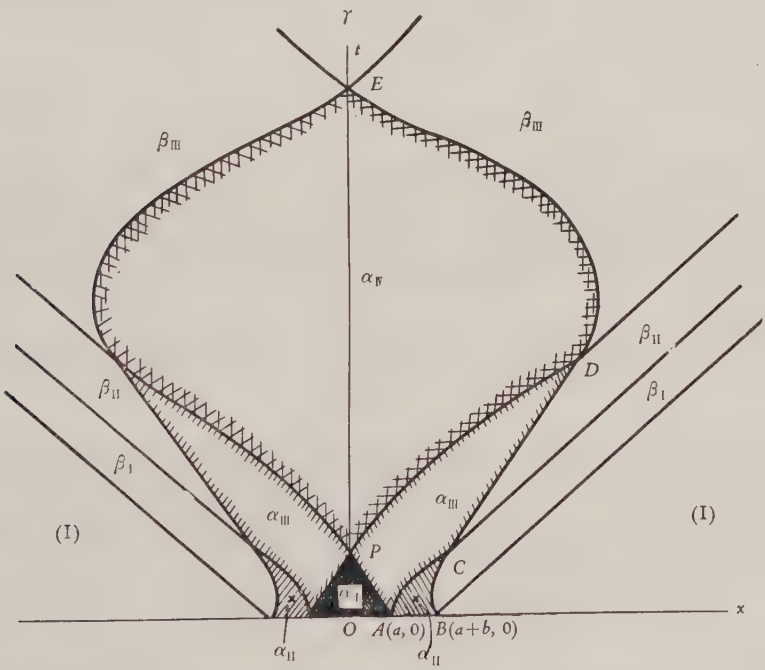

Fig. 4. The region α and β are composed of the four domains, $\alpha_I \sim \alpha_{IV}$ and the three zones $\beta_I \sim \beta_{III}$, respectively. The region γ and (I) is the vacuum.

Employing the same procedure as employed in § 3, (I), we can divide the whole coordinate space into the three regions α, β and γ, which, moreover, are subdivided into a few smaller regions as shown in Fig. 4. Their boundaries and the solutions in these regions are given in the following way.

(1) α_I region

The solution is

$$u=0,\ v=\alpha, \tag{5·6}$$

the boundary line AP is straight, whose gradient $\lambda_\pm$ is given by

$$\lambda_\pm=\pm[(3\alpha^2+2|\varepsilon|)/(\alpha^2+2|\varepsilon|)]^{1/2}. \tag{5·7}$$

(2) α_{II} region

Parametrarizing the coordinate on the x-axis by $\hat{\xi}$, one obtains the following expressions of C and C';

$$\begin{aligned} C(\hat{\xi}) &= v_0^{-1}(\hat{\xi})\ F_0^{-1}(\hat{\xi}), \\ C'(\hat{\xi}) &= v_0^{-1}(\hat{\xi}) F_0^{-1}(\hat{\xi}) \end{aligned} \tag{5·8}$$

in which $F_0(\hat{\xi})$ is defined by

$$F_0(\hat{\xi})=[F(u,\ v)]_{\substack{u=0 \\ v=v_0(\xi)}}$$

From (5.8) and (5.3), u and v can be settled down. Since there holds $u=v$ in the β_I zone, on the boundary curve CB we have

$$\begin{aligned} \lambda_+ &= 1, \\ \lambda_- &= (v^2+\varepsilon)\ (v^2-\varepsilon); \end{aligned}$$

the latter determines the curve CB, showing that at $t=0$, λ_- is equal to -1, as t increases, CB becomes steeper, then parallel to the t-axis for $v=\varepsilon$, and finally, tends to lean rightwards, if we assume that v increases monotonously on CB. This assumption for the variation of v can be justified in the region in which $v_0(\hat{\xi}) \gg \varepsilon$ holds, because on CB we have

$$v=v_0(\hat{\xi})F_0(\hat{\xi})/2F(u=v) \cong \text{const}\cdot v_0(\hat{\xi})\{\text{const}\cdot v_0(\hat{\xi})^{\sqrt{3}-1}+O(\varepsilon)\}.$$

(3) The α_{III} region

This region is of the simple wave type, and the values u and v can be determined from the two equations: $C_0'FY=1$ along the characteristics prolonged from the α_I region and $C(\hat{\xi})FX=1$, prolonged from the α_{II} region. Especially, on the boundary CD, where there holds $u=v$, we have

$$v=(1/2)\,C_0'^{-1}\,F_1^{-1}\ \ (=\text{constant}),$$

hence also

$$\lambda_-=(v^2+|\varepsilon_1|)/(v^2-|\varepsilon_1|)=(1+2|\varepsilon|C_0'^2F_1^2)\ \ (1-2|\varepsilon|C'^2F_1^2)$$

in which F_1 is defined by $[F]_{u=v}$. Therefore, the boundary CD is the straight line. If ε approaches zero, $(C_0'F_1)^2$ tends to zero in the order of $|\varepsilon|^{\sqrt{3}-1}$, hence λ_- tends to zero in the order of $|\varepsilon|^{\sqrt{3}}$. Geometrically speaking, according as ε tends to zero, the curve BC begins immediately to lean towards the right, then C and D leave for infinity (c.f. Fig. 4).

In this limit, $\varepsilon \to 0$, both the regions of β and γ tend to vanish and the α_{IV} region spreads over into the whole upper region. The energy density on the wave front reduces to $\varepsilon\, v^2$ in the limit $b \to 0$. Therefore it tends to zero in the order of $|\varepsilon|^{2-\sqrt{3}}$. This final result coincides with that of the equation discussed in § 4 (I).

It should, however, be emphasized that the point $\varepsilon = 0$ is the algebraic singularity of the solution.

§ 6. Concluding remarks

From the results obtained in this paper, we are discouraged in the attempt to invoke the semi-linear equations to resolve the singularities on the wave front facing the vacuum.

Also, in the quasi-linear equations the circumstances are not improved, so far as the one dimensional propagation is concerned and when we consider such Lagrangians that tend to the linear one in some limit.

However, we have investigated, in this paper, the solutions of the initial value problems which are continuously differentiable, or are to be obtained as the limits of thesegenuine solutions.

On the contrary if the physical causal development is not considered, or in other words if we are not concerned with the initial value problem, one may find other features not obtained here. In fact, the solution obtained by Heisenberg[6] seems to be outside the present frame-work and it may be doubtful whether it corressponds to a solution representing the propagation of the wave under some physical initial condition or not.

As was suggested by Koba,[7] it should also be noted that contrary to Heisenberg's opinion the singularity on the wave front in the nonlinear field has no direct connection with the situation whether or not the Lagrangian depends on a constant, other than mass, with the dimension of length.

The author wishes to express his cordial thanks to Prof. R. Utiyama for his advice.

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Indirect Coupling of Nuclear Spins in Antiferromagnet with Particular Reference to MnF_2 at Very Low Temperatures

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Indirect coupling of nuclear spins through hyperfine interaction with spin waves is discussed in the case of antiferromagnet at very low temperatures. The line width of the F^{19} nuclear magnetic resonance in MnF_2 at 1.4° K observed by Shulman and Jaccarino (~14 oe) proves to come mainly from this coupling. The line width of the Mn^{55} resonance in MnF_2 is also evaluated to be about 600 oe.

§ 1. Introduction

During the past few years, observations of the nuclear magnetic resonance of non-magnetic ions in paramagnetic iron group fluorides have been reported,[1,2,3] and very recently, Shulman and Jaccarino[4] observed the F^{19} nuclear magnetic resonance in the antiferromagnetic state of MnF_2. According to their preliminary report, the resonance at 1.4 K is Gaussian in shape with a width of ~14 oe. This width is too large to be explained from the nuclear dipole interaction, whose contribution is only 6 oe* in the presence of the magnetic field along the c-axis.

As was shown by Moriya[5] and Kranendonk and Bloom[6], the spin lattice relaxation time T_1 becomes longer and longer with decreasing temperature. In actuality, Shulman and Jaccarino found T_1 to be about 20 sec at 4.2 K. In such a temperature region, the thermal motion of the electron magnet is therefore no longer effective for the width of the nuclear spin resonance.

Thus, we are compelled to consider an indirect coupling of nuclear spins through hyperfine interaction to account for the line width mentioned above. The physical meaning of this indirect coupling is the following: a nuclear magnet polarizes the electron spin component transverse to the direction of the sublattice magnetization through hyperfine interaction and another nuclear magnet sees this polarization of the electron magnet again through hyperfine interaction. In the language of spin waves, this process can be stated in the way that the first nucleus virtually excites a spin wave and the second nucleus absorbs it, both through hyperfine interaction. In this way, the two

* Though Shulman and Jaccarino write that the observed width can be explained by the nuclear dipole interaction, it appears that they committed computational errors.

nuclear spins become coupled indirectly. This is similar to Ruderman-Kittel's indirect coupling[7] between nuclei in the metal in which case the conduction electrons play a similar rôle to that of the spin waves in our case. A difference in the two cases is that, while the former indirect coupling is isotropic, the latter one is strongly anisotropic depending on the direction of the axis of the sublattice magnetization.

This coupling will naturally be of a long range character, because if the polarization of the electron magnet were localized in a small region, it would bring about a considerable promotion of the exchange energy. On the other hand, however, there exists an anisotropy energy in the electron spin system, and this energy makes the localized polarization more favourable. Therefore the range of the coupling is determined by the competition between the exchange energy and the anisotropy energy. In actual case of Mn F_2, the range will extend over a few atomic distances.

In his preliminary report, Suhl[8] presented a theory of the indirect coupling of nuclear spins in ferromagnet. His mechanism is essentially the same as ours, but the line width formula given by him is not very accurate. The present study was done independently of him. We shall calculate the line width of the F^{19} nuclear magnetic resonance, as well as that of Mn^{55} although Suhl also gives a rough estimate of the latter.

§ 2. Indirect coupling between nuclear spins and the line width of nuclear resonance of magnetic ions due to it

Our problem is to discuss the electron spin system perturbed by the hyperfine interaction, whose perturbation depends on the nuclear spin state. We shall study this problem by the use of the spin wave technique[9]. For the sake of simplicity, let us consider a case where we can divide the magnetic lattice into two sublattices in such a way that each lattice point belonging to one sublattice of up spins is surrounded by lattice points belonging to the other sublattice of down spins. Let us denote these two kinds of lattice points by j and k, respectively. We shall further introduce creation operators of spin deviation, a_j^* and b_k^*, by $S_j^z = S - a_j^* a_j$ and $S_k^z = -S + b_k^* b_k$, in which $\boldsymbol{S}_j(S_j^x, S_j^y, S_j^z)$ is the spin operator of the j-th lattice point. Then, $S_j^{\pm} = S_j^x \pm iS_j^y$ and $S_k^{\pm} = S_k^x \pm iS_k^y$ are approximately written as $S_j^+ = \sqrt{2S}\, a_j$, $S_j^- = \sqrt{2S}\, a_j^*$, $S_k^- = \sqrt{2S}\, b_k^*$ and $S_k^+ = \sqrt{2S}\, b_k$, respectively.

We shall take the unperturbed Hamiltonian of the electron spin system as

$$H_0 = 2J \sum_{<jk>} \boldsymbol{S}_j \cdot \boldsymbol{S}_k - 1/2 \cdot \left(\sum_j (S_j^z)^2 + \sum_k (S_k^z)^2 \right), \tag{1}$$

where J denotes the exchange integral reversed in sign and K the anisotropy constant. If we rewrite (1) in terms of the spin creation and annihilation operators, the terms of order S will represent the spin wave Hamiltonian, which will be taken anew as our approximate unperturbed Hamiltonian. Let us then introduce the Fourier transformed annihilation and creation operators by

$$a_\lambda=\sqrt{\frac{1}{N}}\sum_j a_j \exp\ (-i\,\lambda\, j), \tag{2}$$

$$b_\lambda=\sqrt{\frac{1}{N}}\sum_k b_k \exp\ (i\,\lambda\, k)$$

and their complex conjugates. Here N denotes the number of magnetic lattice points belonging to one sublattice, λ a wave vector, and λj or λk a scalor product between λ and lattice vector j or k. Then we have

$$H_0=2JS\sum_\lambda [\gamma_0(1+\Delta^2/2)\ (a_\lambda^*\ a_\lambda+b_\lambda^* b_\lambda)$$
$$+\gamma_\lambda(a_\lambda\ b_\lambda+a_\lambda^*\ b_\lambda^*)\,], \tag{3}$$

where

$$\gamma_\lambda=\sum_\rho \exp\ (i\,\lambda\,\rho), \tag{4}$$

ρ being a vector drawn from a magnetic atom to the nearest neighbouring magnetic atoms, and

$$\Delta^2=K/\gamma_0\ J. \tag{5}$$

The diagonalization of (3) is well-known. The result is as follows:

$$H_0=\sum_\lambda \hbar\,\omega_\lambda(\alpha_\lambda^*\ \alpha_\lambda+\beta_\lambda^*\ \beta_\lambda), \tag{6}$$

where

$$\hbar\omega_\lambda=2\gamma_0\ JS\sqrt{1-(\gamma_\lambda/\gamma_0)^2+\Delta^2}\ ;\ \Delta^2\ll 1, \tag{6a}$$

and new coordinates α_λ and β_λ satisfying commutation relations $[\alpha_\lambda,\ \alpha_{\lambda'}^*]=[\beta_\lambda,\ \beta_{\lambda'}^*]$ $=\delta_{\lambda\lambda'}$, $[\alpha_\lambda,\ \beta_{\lambda'}]=[\alpha_\lambda^*,\ \beta_{\lambda'}^*]=0$ etc. are defined by

$$a_\lambda=\cosh\ \theta_\lambda\ \alpha_\lambda+\sinh\ \theta_\lambda\ \beta_\lambda^*,$$
$$b_\lambda=\sinh\ \theta_\lambda\ \alpha_\lambda^*+\cosh\ \theta_\lambda\ \beta_\lambda \tag{7}$$

with

$$\tanh\ 2\theta_\lambda=-\gamma_\lambda/\gamma_0[1+\Delta^2/2]. \tag{7a}$$

Let us look for the indirect coupling of nuclear spins which belong to the magnetic atoms. Our perturbing Hamiltonian arising from the hyperfine interaction is then written

$$H'=A\sum_n \boldsymbol{I}_n\cdot\boldsymbol{S}_n, \tag{8}$$

where A denotes the hyperfine coupling constant, which is assumed to be isotropic, and $\boldsymbol{I}_n$ the spin operator of the nucleus of the n-th magnetic atom. The z-component of (8) is almost ineffective to polarize the electron spin longitudinal to the direction of sublattice magnetization at very low temperatures. Confining our attention to the transverse component only, we write (8) in terms of the normal coordinates defined by (2)

and (7), and obtain

$$H'=\sqrt{\frac{S}{2N}}A\sum_{\lambda}[(\cosh\theta_\lambda I_\lambda^-+\sinh\theta_\lambda J_\lambda^-)\alpha_\lambda$$
$$+(\sinh\theta_\lambda I_\lambda^++\cosh\theta_\lambda J_\lambda^+)\beta_\lambda]+\text{c.c.}, \tag{9}$$

where

$$I_\lambda^+=\sum_j I_j^+\exp(-i\lambda j),\quad J_\lambda^-=\sum_k I_k^-\exp(i\lambda k) \tag{10}$$

are the Fourier transformed operators of $I^\pm=I^x\pm iI^y$.

The second order perturbation of (9) gives

$$\Delta E=-\frac{1}{2}A^2S\frac{1}{N}\sum_\lambda\frac{1}{\hbar\omega_\lambda}\Big[\frac{1}{2}\cosh 2\theta_\lambda(I_\lambda^+I_\lambda^-+I_\lambda^-I_\lambda^+$$
$$+J_\lambda^+J_\lambda^-+J_\lambda^-J_\lambda^+)+\sinh 2\theta_\lambda(I_\lambda^+J_\lambda^-+I_\lambda^-J_\lambda^+)$$
$$-\sum_j I_j^z+\sum_k I_k^z\Big], \tag{11}$$

whence an effective Hamiltonian of nuclear spin system follows:

$$H_{\text{eff}}=1/2\cdot D[\sum_j(I_j^z)^2+\sum_k(I_k^z)^2]$$
$$-1/2\cdot\sum_{j>j'}B_{jj'}(I_j^+I_{j'}^-+I_j^-I_{j'}^+)$$
$$-1/2\cdot\sum_{k>k'}B_{kk'}(I_k^+I_{k'}^-+I_k^-I_{k'}^+)$$
$$-\sum_{j,k}C_{jk}(I_j^+I_k^-+I_j^-I_k^+). \tag{12}$$

Here D, $B_{jj'}$ and C_{jk} are respectively written, using (6a) and (7a), as

$$D=\frac{A^2}{2\gamma_0J}\cdot\frac{1}{N}\sum_\lambda\frac{1}{[1-(\gamma_\lambda/\gamma_0)^2+\Delta^2]}, \tag{13a}$$

$$B_{jj'}=\frac{A^2}{2\gamma_0J}\cdot\frac{1}{N}\sum_\lambda\frac{\cos\lambda(j-j')}{[1-(\gamma_\lambda/\gamma_0)^2+\Delta^2]}, \tag{13b}$$

$$C_{jk}=\frac{A^2}{2\gamma_0J}\cdot\frac{1}{N}\sum_\lambda\frac{(\gamma_\lambda/\gamma_0)\cos\lambda(j-k)}{[1-(\gamma_\lambda/\gamma_0)^2+\Delta^2]}. \tag{13c}$$

In (12) we neglected terms leading to shift of the resonance frequency, because they are small compared with the corresponding terms arising from the longitudinal part of (8).

It will be interesting to look into the range of the indirect coupling thus obtained. Namely, we shall find an asymptotic expression of $B_{jj'}$, for large $r\equiv j-j'$. If we put

$$f(r)\equiv\frac{1}{N}\sum_\lambda\frac{\cos\lambda r}{[1-(\gamma_\lambda/\gamma_0)^2+\Delta^2]}, \tag{14}$$

the contribution of the summation over λ to $f(r)$ for large r will predominantly come from a region of small λ. So we may substitute

$$\gamma_\lambda/\gamma_0 \simeq 1-\frac{\alpha}{2}\lambda^2$$

into (14), where α is equal to $1/3$ for the simple cubic lattice and to $1/4$ for the body-centered cubic lattice. Thus we get

$$f(r) \simeq \frac{1}{4\pi\alpha}\cdot\frac{1}{r}\exp(-\kappa r), \tag{15}$$

in which $\kappa = 1/\sqrt{\alpha}$. This equation is essentially the same as that given by Suhl.

However, it will not be satisfactory for the calculation of the actual line width to use (15), because the contribution to line width must come from the nuclear spins which are within a range of $1/\kappa$ and the asymptotic expression (15) breaks down.

Let us then find an expression for the line width from (12). Now, a nuclear spin belonging to one of the sublattices is subjected to a local field which is different from that belonging to the other sublattice and hence $I_j^+ I_{j'}^-$ must oscillate with a high frequency. Accordingly, $k(j)$-th nuclear spin will not induce any important fluctuating field on the $j(k)$-th nuclear spin in a Larmour cycle of the latter. This situation is completely the same as that in Van Vleck's theory of the exchange broadening for nuclei with two different kinds of moment.[10]

Thus, our Hamiltonian for the line width effectively reduces to

$$H_{\text{eff}} = -\sum_{j>j'} B_{jj'}(\boldsymbol{I}_j\cdot\boldsymbol{I}_{j'} - I_j^z I_{j'}^z) + 1/2\cdot D\sum_j (I_j^z)^2. \tag{16}$$

The second part in the right-hand side of this equation is of the quadrupole type, which is effective for the nucleus of spin quantum number larger than one half and is comparable in magnitude with the first part. If we neglect this quadrupolar term for the moment, the second moment formula of Van Vleck leads to

$$<(\hbar\Delta\omega)^2>_{\text{AV}} = (1/3)\,I(I+1)\sum_{j'} B_{jj'}{}^2. \tag{17}$$

Substituting (13b) for $B_{jj'}$, and using

$$\sum_{j'}\cos\lambda(j-j')\cos\lambda'(j-j') = N\delta_{\lambda\lambda'}, \tag{18}$$

we have

$$<(\hbar\Delta\omega)^2>_{\text{AV}} = (1/3)\,I(I+1)\left(\frac{A^2}{2\gamma_0 J}\right)^2 f, \tag{19}$$

$$f = \frac{1}{N}\sum_\lambda \frac{1}{[1-(\gamma_\lambda/\gamma_0)^2+\Delta^2]^2}. \tag{19a}$$

§ 3. Line width of the F^{19} nuclear magnetic resonance in MnF_2

The line width of the F^{19} resonance at very low temperatures can be calculated in the same way as in the preceding section. Indirect coupling in this case occures by the following mechanisms:

(1) Combined actions of $Mn^{55}-Mn^{++}$ and $Mn^{++}-F^{19}$ hyperfine interactions. (Mn^{55} and F^{19} stand for the respective nuclei and Mn^{++} for the electronic spin.)
(2) Combined actions of $Mn^{55}-Mn^{++}$ hyperfine and $F^{19}-Mn^{++}$ nuclear-electronic dipolar interactions.
(3) Combined actions of $F^{19}-Mn^{++}-F^{19}$ hyperfine interactions. Both processes (1) and (2) lead to $Mn^{55}-F^{19}$ indirect coupling, and (3) to $F^{19}-F^{19}$ indirect coupling. In any case, however, the coupling is of the form $I_j^- I_n^-$, and hence the coupling between nuclear spins with different local fields or different magnetic moments cannot give any fluctuation which leads to line width.*)

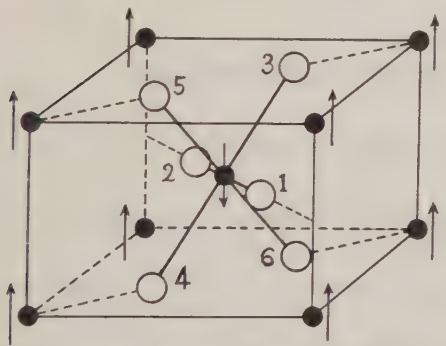

Fig. 1. Crystal structure of MnF_2 with $a=4.8734$A, $c=3.3103$A and $u=0.310$ (ref. 18).

Let us now confine our attention to the coupling arising from the third process. The crystal structure of MnF_2 is of the rutile type as shown in Fig. 1 with the magnetic superstructure found by Erickson and Shull[11] is also shown.

In this figure, F^- designated by 1 is surrounded by three Mn^{++}'s. It has been shown by Tinkham[12], in his paramagnetic resonance experiment of Mn^{++} diluted in ZnF_2, that the hyperfine coupling constant between these three Mn^{++} ions and the central F^- are approximately equal in magnitude. It has also been pointed out by Bleaney[13] that Tinkham's experiment is consistent with the nuclear resonance of F^{19} in MnF_2. Thus, F^{19} 1 is subjected to a local field arising from two up spins and one down spin, and hence the net local field is from one up spin. The situation is the same for F^- 2. On the other hand, those designated by 3,4,5 and 6 are each surrounded by two down spins and one up spin and hence these are all subjected to the net local field from one down spin. We are therefore to look for the indirect coupling among nuclear spins of the same type, such as 1 and 2. These F^{19} of the same type are on the set of planes parallel to (1 −1 0). One of these planes is drawn in Fig. 2, where the j-th F^- is surrounded by the j_1-, j_2- and j_3-th Mn^{++}'s. Assuming the $F^{19}-Mn^{++}$ hyperfine interaction to be isotropic, we can write our perturbing Hamiltonian as

$$H'=A\sum_j \boldsymbol{I}_j\cdot(\boldsymbol{S}_{j1}+\boldsymbol{S}_{j2}+\boldsymbol{S}_{j3}). \tag{20}$$

* In the second mechanism, we shall obtain the coupling of the forms $I_j^+ I_n^z$, $I_j^+ I_n^+$ and their complex conjugates, but the conclusion remains unaltered.

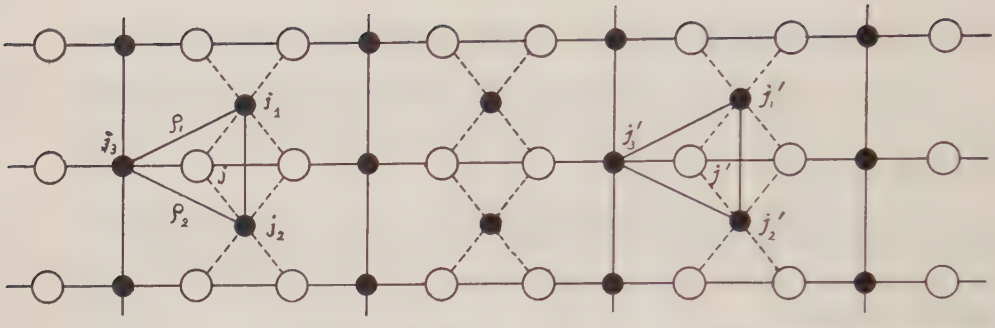

Fig. 2. The ion arrangements on a plane parallel to (1 −1 0). The F^- ion is designated by open circle and the Mn^{++} ion by black circle which includes the corner ion with down spin and the centered ion with up spin. A F^- ion is surrounded by three Mn^{++}'s forming a triangle. Note that a corner ion belongs to two triangles.

Let ρ_1 and ρ_2 be vectors defined by

$$j_1-j_3\equiv\rho_1,\quad j_2-j_3\equiv\rho_2. \tag{21}$$

Resolution of the transverse component of (19) into normal coordinates gives

$$H'=\sqrt{\frac{S}{2N}}A\sum_{\lambda}\Big[\{(\exp[i\lambda\rho_1]+\exp[i\lambda\rho_2])\cosh\theta_\lambda+\sinh\theta_\lambda\}$$
$$\times(\sum_j I_j^-\exp[i\lambda j_3])\alpha_\lambda+\{(\exp[-i\lambda\rho_1]+\exp[-i\lambda\rho_2])\sinh\theta_\lambda$$
$$+\cosh\theta_\lambda\}(\sum_j I_j^+\exp[-i\lambda j_3])\beta_\lambda\Big]+\text{c.c.} \tag{22}$$

The indirect coupling between nuclear spins can be obtained in the same way as in the preceding section:

$$H_{\text{eff}}=-(1/2)\sum_{j>j'}D_{jj'}(I_j^+I_{j'}^-+I_j^-I_{j'}^+), \tag{23}$$

$$D_{jj'}=A^2S\cdot\frac{1}{N}\sum_{\lambda}\frac{\cos\lambda(j_3-j_3')}{\hbar\omega_\lambda}[\{3+2\cos\lambda(\rho_1-\rho_2)\}$$
$$\times\cosh 2\theta_\lambda+2(\cos\lambda\rho_1+\cos\lambda\rho_2)\sinh 2\theta_\lambda]. \tag{23a}$$

With the help of (18), the second moment is obtained from (23) as

$$<(\hbar\Delta\omega)^2>_{\text{AV}}=\frac{2}{3}I(I+1)\left(\frac{A^2}{2\gamma_0 J}\right)^2 f'. \tag{24}$$

In applying (18), the summation over j' in (23) is replaced by that over j_3' running over the Mn^{++} sites with down spin, in which a lattice point j_3' belongs to two triangles (cf. Fig. 2), and hence a factor of 2 should be taken into account. f' is defined by

$$f'=\frac{1}{N}\sum_{\lambda}\frac{1}{[1-(\gamma_\lambda/\gamma_0)^2+\Delta^2]^2}[\{3+2\cos\lambda(\rho_1-\rho_2)\}$$
$$+2(\cos\lambda\rho_1+\cos\lambda\rho_2)(\gamma_\lambda/\gamma_0)]^2. \tag{24a}$$

Denoting the components of λ by λ_1, λ_2 and λ_3, we have

$$\cos\lambda(\rho_1-\rho_2)=\cos\lambda_3,\ \cos\lambda\rho_1=\cos\{1/2\cdot(\lambda_1+\lambda_2+\lambda_3)\},$$
$$\cos\lambda\rho_2=\cos\{1/2\cdot(\lambda_1+\lambda_2-\lambda_3)\}. \tag{25}$$

§ 4. Numerical evaluation of the line width.

1) *Evaluation of integrals* $f(0)$, f and f'.

It is necessary to evaluate f and f' defined respectively by (19a) and (24a). In the body-centered cubic lattice, we have

$$\gamma_\lambda/\gamma_0=\cos\frac{\lambda_1}{2}\cos\frac{\lambda_2}{2}\cos\frac{\lambda_3}{2}. \tag{26}$$

With this form factor, $f(0)$ defined by (14) is evaluated as follows:

$$f(0)=\frac{1}{(2\pi)^3}\int_0^{2\pi}\!\!\int\!\!\int\frac{d\lambda_1\ d\lambda_2\ d\lambda_3}{1-(\gamma_\lambda/\gamma_0)^2+\Delta^2}$$
$$=\frac{2}{\pi^2}\int_0^\pi F\left(\frac{\pi}{2},\ \frac{1}{\sqrt{1+\Delta^2}}\cos\frac{\lambda}{2}\right)d\lambda, \tag{27}$$

where $F\left(\frac{\pi}{2},\ k\right)$ is a complete elliptic integral of the first kind with modulus k. In the same way,

$$f=F_0+E_0, \tag{28}$$
$$f'=32-103F_0-40F_2+16F_4$$
$$+41E_0+56E_2+16E_4-32G, \tag{29}$$

where we neglected higher order terms with respect to Δ^2, on the assumption that $\Delta^2\ll 1$. F_n, E_n and G are defined by the following:

$$F_n=\frac{1}{\pi^2}\int_0^\pi F\left(\frac{\pi}{2},\ \frac{1}{\sqrt{1+\Delta^2}}\cos\frac{\lambda}{2}\right)\cos^n\frac{\lambda}{2}d\lambda, \tag{30a}$$
$$E_n=\frac{1}{\pi^2}\int_0^\pi E\left(\frac{\pi}{2},\ \frac{1}{\sqrt{1+\Delta^2}}\cos\frac{\lambda}{2}\right)\left[1-\frac{1}{1+\Delta^2}\cos^2\frac{\lambda}{2}\right]^{-1}$$
$$\times\cos^n\frac{\lambda}{2}d\lambda, \tag{30b}$$
$$G=\frac{1}{\pi^2}\int_0^\pi\left[\pi-3F\left(\frac{\pi}{2},\ \frac{1}{\sqrt{1+\Delta^2}}\cos\frac{\lambda}{2}\right)\right.$$
$$\left.+\left\{1-\frac{1}{1+\Delta^2}\cos^2\frac{\lambda}{2}\right\}^{-1}E\left(\frac{\pi}{2},\ \frac{1}{\sqrt{1+\Delta^2}}\cos\frac{\lambda}{2}\right)\sec^2\frac{\lambda}{2}d\lambda. \tag{30c}\right.$$

$E\left(\frac{\pi}{2}, k\right)$ is a complete elliptic integral of the second kind with modulus k. Evaluation of (27), (28), and (29) were performed by numerical integration for three values of Δ. The result is given in Table 1.

As Δ decreases, F_n tends to a finite value. The limiting value of F_0 has been estimated to be 1.3932 by Watson.[14] Since E_n increases with decreasing Δ, we can neglect F_n for very small Δ. Further, some inspections tell us $G \sim E_0 \sim 1 - \pi\Delta$ for $\Delta \ll 1$, giving asymptotic expressions for f and f'

$$f \cong 1/\pi\Delta, \quad f' \cong 81\, f. \tag{31}$$

But, these expressions are unable to give accurate enough values of f and especially f' in the region of $\Delta \gtrsim 0.2$.

2) *Line width of the* F^{19} *resonance in* $Mn(F^{19})_2$.

As was pointed out, the $F^{19}-Mn^{++}$ hyperfine interaction is isotropic and three couplings of a central F^{19} with its surrounding Mn^{++}'s are nearly equal in magnitude. Based on Tinkham's study, Shulman and Jaccarino[15] estimated A to be about 16×10^{-4} cm^{-1}, excluding dipolar contribution.

An ambiguity might arise from our estimate of the exchange integral J. Since there is an evidence that the super-exchange interaction between two neighbouring Mn^{++} along the c-axis is approximately equal to that between two neighbouring ions along the body-diagonal direction,[16] we may assume the number of neighbours effective for the Néel temperature to be $\gamma_0 - 2$. Using $kT_N = (2/3)(\gamma_0 - 2)JS(S+1)$ with $T_N = 67°K$ (Stout-Adams[17]), we estimate $\gamma_0 J$ to be 2.1×10^{-15} erg.

Table I. Numerical values of $f(0)$, f and f' computed for three values of Δ.

Δ	$f(0)$	f	f'
0.100	1.33	3.94	216
0.179	1.28	2.65	118
0.224	1.27	2.33	93

We have next to estimate Δ defined by (5). The magnetic anisotropy of MnF_2 has been studied experimentally in great detail by Stout and Griffel.[18] From their result Keffer[19] has concluded that the magnetic anisotropy of MnF_2 dominantly come from magnetic dipolar interaction. His conclusion might appear to modify drastically our unperturbed Hamiltonian. It cannot be so actually, since the magnetic dipolar interaction in antiferromagnet proves to bring about nothing but the anisotropy term similar to that introduced somewhat phenomenologically in the present paper.[20] Our final result will therefore not suffer from any appreciable modification by introducing a more realistic Hamiltonian. Keffer estimated the effective exchange field H_E at 0 K to be 5.4×10^5 oe from perpendicular magnetic susceptibility and the effective anisotropy field H_A to be 8.8×10^3 oe. These estimates give the antiferromagnetic resonance frequency which is in good agreement with that observed, i.e. within ten percents.[21] As easily shown, $H_E = 2\gamma_0 JS/\gamma\hbar_0$ and $H_A = KS/\gamma\hbar_0$, and hence we have $\Delta = (1/2)(H_A/H_E)^{1/2} = 0.179$. It may be noted that $\gamma_0 J$ estimated from Keffer's H_E is in agreement within ten percents with that derived from the Néel temperature T_N.

The second moment of F^{19} resonance line is finally estimated to be $\langle(\hbar\Delta\omega)^2\rangle_{AV}^{1/2}=1.8\times10^{-22}$ erg. It corresponds to a maximum slope separation of the absorption curve of 13.6 oe ($\gamma_{F^{19}}\hbar=2.655\times10^{-23}$ e.m.u.), which is in excellent agreement with the observed line width of $\sim$14 oe. Perhaps this agreement is rather fortuitous, because our spin wave Hamiltonian is not realistic in view of the presence of an exchange interaction between two neighbouring magnetic ions along the c-axis besides that between neighbours along the body-diagonals. But the order of magnitude may be sure.

As is pointed out in § 1, the nuclear dipole interaction gives only a small contribution to $\langle(\hbar\Delta\omega)^2\rangle$. In the presence of a static magnetic field along the c-axis, a calculation gives

$$\langle(\hbar\Delta\omega)^2\rangle_{AV}=3.1\times10^{-45}\ \text{erg}^2 \quad \text{for } F^{19}-F^{19} \text{ (from the homo-type } F^{19})\ *)$$
$$0.1\times10^{-45} \quad ,, \quad F^{19}-F^{19} \text{ (from the hetero-type } F^{19})$$
$$3.0\times10^{-45} \quad ,, \quad F^{19}-Mn^{55}.$$

Here we may note that the contribution coming from the hetero-type F^{19} and Mn^{55} nuclear spins is to be calculated with the use of the Van Vleck formula with different magnetic moments. This leads to a reduction by a factor $(2/3)^2$ and we see that the nuclear dipole interaction contributes to $\langle(\hbar\Delta)\omega^2\rangle$ only about one fifth of that computed from the indirect coupling.

3) *Line width of the* Mn^{55} *resonance in* $Mn^{55}F_2$.

The nuclear spin of Mn^{55} strongly interacts with its host electron spin. According to Tinkham, the A-value is 96 oe in units of $g\mu_0$, which corresponds to $9.0\times10^{-3}\text{cm}^{-1}$. Using (19), (19a) and Table 1, we have $\langle(\hbar\Delta\omega)^2\rangle^{1/2}=2.1\times10^{-21}$ erg. It is larger than the corresponding value for F^{19} by a factor of 10. The maximum slope separation of the absorption curve for Mn^{55} resonance is thus estimated to be about 600 oe ($\gamma_{Mn^{55}}\hbar=0.70\times10^{-23}$ e.m.u.). If we take into account the part of the quadrupole type, the line width become larger by about ten percents.**)

The large predicted width of the Mn^{55} line would make it impossible to detect the Mn^{55} resonance. The large difference for the Mn^{55} width from that of F^{19} comes mainly from that the hyperfine constant A of $Mn^{++}-Mn^{55}$ amounts to six times as large as that of $Mn^{++}-F^{19}$.

Although we have yet no observation of the NMR of the magnetic ions in antiferromagnet, we may give here the formula of the second moment for the case where $J\lesssim1$. It is written

$$\langle(\hbar\Delta\omega)^2\rangle_{AV}\simeq\frac{1}{3\pi}I(I+1)\left(\frac{A^2}{\hbar\omega_0}\right)\cdot\left(\frac{A^2S}{2\gamma_0 J}\right),$$

* The writer is indebted to Mr. O. Nagai for his some inspections.

** The value of $\langle(\hbar\Delta\omega)^2\rangle$ arising from the quadrupolar part is

$$(1/10)\,(2I-1)\,(I+3/2)\,(A^2/2\gamma_0 J)^2 f^2(0)\sim1.4\times10^{-42}\ \text{erg}^2.$$

where ω_0 denotes the antiferromagnetic resonance frequency when there is no static magnetic field : $\hbar\omega_0 = 2S\sqrt{\gamma_0 JK}$.

§ 5. Conclusion

In Table 2, our numerical results are tabulated with some experimental values used. Throughout the present paper, we confined our study to the region of temperatures far below the Neel point. With increasing temperature, the neglected longitudinal part would become effective and the indirect coupling would approach to being isotropic. in the form $\boldsymbol{I}_j \cdot \boldsymbol{I}_k$. Then, the contribution from the homotype nuclear spins to the line width would diminish on the one hand, and that from the hetero-type nuclear spins increase on the other hand. In the temperature region where such effects become appreciable, however, the spin-lattice relaxation time would be sufficiently short and would cover the effect of the indirect coupling.

Table. 2. Summary of the results with some numerical data used.

$A(Mn^{++}-F^{19})$	3.2×10^{-19} erg
$A(Mn^{++}-Mn^{55})$	1.8×10^{-18} erg
$\gamma_0 J$	2.1×10^{-15} erg
Δ	0.179
$\langle(\Delta H)^2\rangle_{AV}^{1/2}(F^{19})_{calc.}$	6.8 oe
$\langle(\Delta H)^2\rangle_{AV}^{1/2}(F^{19})_{obs.}$	~7.0 oe
$\langle(\Delta H)^2\rangle_{AV}^{1/2}(Mn^{55})_{calc.}$	~300 oe

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Quasi-Chemical Equilibrium Theory, Part II

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The quasi-chemical equilibrium, or pair correlation, approximation to statistical mechanics is written in terms of second-quantization formalism. This involves an entension of the original theory to include an Ansatz for off-diagonal elements of the density matrix. The formal expressions involve "labelling operators" acting in a purely formal Hilbert space. These labelling operators obey Bose commutation rules for correlations between even numbers of particles (e.g. pair correlations), Fermi commutation rules for correlations between odd numbers of particles. The labelling operators provide an algebraic way of formulating the restriction to certain types of "graphs". We also show how to include higher-order correlations into the Ansatz.

The main results of the original theory can be derived more rapidly with the new formalism. However, certain correction terms obtained earlier are shown to be in error, necessitating a reinvestigation of the nature of the condensation phenomenon. This re-investigation is carried out in two following papers, and proves to lead to no essential modifications of the previous results.

The correlation matrices which enter into the Ansatz need to be related to the Hamiltonian of the system, and to the thermodynamic variables which define its thermodynamic state. A variational formulation is developed for this purpose, but no explicit calculations are carried out in this paper.

§ 1. Introduction

The essence of the quasi-chemical equilibrium approximation[1] is the replacement of the true (not normalized) density matrix

$$\mathcal{U} = \exp(\beta\mu N - \beta H) \tag{1·1}$$

by a certain definite *functional form* which is built up from two basic operators:

1) A single-particle Ursell matrix $\langle k|U_1|k'\rangle$,

2) A pair-correlation Ursell matrix $\langle k_1 k_2|U_2|k_1' k_2'\rangle$.

The functional form chosen for $\mathcal{U}$ was given implicitly in reference 1), by writing down a definite expression for the trace of the matrix $\mathcal{U}$ in terms of the basic Ursell

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matrices U_1 and U_2. This trace defines the grand canonical potential Ω of the system through the relation

$$\exp(-\beta\Omega) = \text{Trace } \mathcal{U}. \tag{1·2}$$

The remainder of reference 1) was devoted to carrying out the reductions necessary to obtain a useable expression for (1·2) in terms of properties of the basic matrices U_1 and U_2. Under certain reasonable assumptions about these properties, it was shown that one may expect a condensation phenomenon similar to Bose-Einstein condensation of the "quasi-molecules" which appear naturally in the formalism (as eigenfunctions of a matrix U_2 related to U_2). It was conjectured that this condensation does in fact occur in some metals, and is responsible for the transition to the superconducting state.

This paper is devoted to improving and elucidating the formalism in several ways:

(1) The definition of the functional form chosen for $\mathcal{U}$ in terms of U_1 and U_2 was incomplete, because only the diagonal elements of the operator $\mathcal{U}$ enter into the trace. We are therefore free to choose the off-diagonal elements of $\mathcal{U}$, and each such choice defines an "extension" of the quasi-chemical equilibrium theory. In this paper, we give such an extension, which allows particularly simple formal expressions and is, we feel, a "natural" extension.

(2) The restriction to pair correlations only is probably quite sufficient in most cases; after all, the much more extreme *independent particle model*, which results from this theory by setting $U_2=0$, is known to describe most metallic properties very well indeed. Nevertheless, it seems desirable to show, at least in principle, just how higher order Ursell matrices U_3, U_4, $\cdots$ can be taken into account by a natural extension of the formalism. This is done in section 5 of this paper.

(3) The simple formalism developed here allows a quick rederivation of the results of reference 1). This is done in sections 3 and 4, with the aid of a formalism developed by Dyson[2] for the discussion of spin waves. We recover the results of references 1), with one exception: the higher order correction terms given in Appendix II of that reference are in error, due to a wrong combinatorial factor. This makes no difference above the transition point, but it does necessitate a re-investigation of the nature of the transition. This re-investigation is the subject of subsequent papers in this series. It suffices to state here that the essential conclusions are unchanged: the transition is in all respects similar to the ordinary Bose-Einstein condensation.

(4) In order to define the approximation completely, it is of course necessary to give some prescription for constructing the Ursell matrices U_1 and U_2, given the Hamiltonian of the system, the chemical potential μ, and the temperature $T=(k\beta)^{-1}$. The prescription given in reference 1) was the simplest possible choice, defined as follows: Let H_1 be the part of the Hamiltonian H in the one-particle space (H itself is given in second-quantized form, for any number of particles). Similarly, H_2 is the projection of H onto the 2-particle space. Then U_1 and U_2 were defined by:

$$U_1 = \exp(-\beta H_1) \tag{1·3}$$

and

$$U_2 = \exp(-\beta H_2) - \exp(\beta H_1) \times \exp(-\beta H_1) \qquad (1\cdot 4)$$

where the cross $\times$ denotes the direct product of the two single-particle operators. If we choose the states $|k\rangle$ so that U_1 is diagonal (this is not a restriction in principle), the matrix elements are

$$\langle k|U_1|k'\rangle = \delta_{kk'} \exp(-\beta\varepsilon_k) \qquad (1\cdot 3')$$

$$\langle k_1 k_2|U_2|k_1' k_2'\rangle = \langle k_1 k_2|\exp(-\beta H_2)\,|k_1' k_2'\rangle - \delta_{k_1 k_1'}\delta_{k_2 k_2'} \exp[-\beta(\varepsilon_{k_1}+\varepsilon_{k_2})]\,. \qquad (1\cdot 4')$$

However, there is no reason to believe that the "best" single-particle Hamiltonian in the many-body system is just the Hamiltonian H_1 of a true single particle, without any other particles around. In other words, we expect to find an "effective mass" or even more complicates expressions for the "best" H_1, in analogy to the work of Brueckner and other[3]. Indeed, the theory should reduce to the usual approximations for the ground state of the many-body system (Hartree-Fock, Brueckner, or extensions thereof), when we go to the limit of zero temperature. At finite temperatures, we expect our "best" H_1 and H_2 to depend not only on the density of particles, but also on the temperature itself.

It has been found[4] that the choice $(1\cdot 3)$, $(1\cdot 4)$ does indeed lead to physically unreasonable results. A better method of specifying U_1 and U_2 is therefore needed. In this paper we show how this can be done in principle, by means of a variational approach. In the case of the independent particle approximation ($U_2 = 0$) the variational calculation has been carried through completely by Husimi[5], and provides a natural extension of the Hartree-Fock method to statistical mechanics. The corresponding calculation for the quasi-chemical equilibrium theory is much more complicated, and has not been completed as yet.

It should be noted that the states labelled "k" in this formalism need not be bare-particle states, but can be thought of as effective-particle states. It is well known[3] that a considerable fraction of the effect of actual correlations between the bare particles can be taken into account by altering the zero-order Hamiltonian of the system. This should of course be done *before* introducing the pair correlation terms U_2 into the density matrix of the system. That is, the U_2 matrix is meant to encompass those additional effects of pair correlations which are impossible to include by *any* independent particle model.

§ 2. The functional form of the density operator

The diagonal elements of $\mathcal{U}$ were defined implicity by equation $(3\cdot 7)$ of reference 1). We write, as usual:

$$\text{Trace } \mathcal{U} = \sum_N z^N \exp(-\beta F_N), \qquad (2\cdot 1)$$

where the "activity" z is related to the chemical potential μ by

$$z = \exp(\beta\mu) \qquad (2\cdot 2)$$

and $\exp(-\beta F_N)$ is given by equation $(3\cdot 7)$ of reference 1). We can rewrite the result

quite naturally as a sum of terms, each term giving the contribution of a combination of N_1 "single particles" and N_2 "pair correlations", as follows* :

$$\begin{aligned}\mathrm{Trace}\,\mathcal{U} = \sum_{N_1}\sum_{N_2}\sum_{k_1\cdots k_{N_1}}\sum_{m_1\cdots m_{2N_2}}\sum_{P}\frac{(-1)^P}{N_1!N_2!2^{N_2}}\\ \times\langle k_1|zU_1|k_1\rangle\langle k_2|zU_1|k_2^P\rangle\cdots\langle k_{N_1}|zU_1|k_{N_1}^P\rangle\\ \times\langle m_1m_2|z^2U_2|m_1^Pm_2^P\rangle\cdots\langle m_{2N_2-1}m_{2N_2}|z^2U_2|m_{2N_2-1}^Pm_{2N_2}^P\rangle. \qquad (2\cdot3)\end{aligned}$$

The permutations P transform the set of N_1+2N_2 indices $k_1,\cdots,m_{2N_2}$ into the permuted indices $k_1',\cdots,m_{2N_2}'$. All these permutations are merely an expression of the exclusion principle, and there is a much neater formalism, that of second quantization, to deal with systems obeying the exclusion principle. In this section, we shall therefore attempt to construct an operator $\mathcal{U}$ in terms of second-quantized space, such that the trace of $\mathcal{U}$ is indeed equal to (2·3).

We do not lose anything in generality, and we gain in simplicity, by choosing the basic states $|k\rangle$ so that U_1 is diagonal. Since the activity z always appears coupled with U_1, we define the set of numbers u_k by

$$\langle k|zU_1|k'\rangle=\delta_{kk'}u_k\,. \qquad (2\cdot4)$$

The u_k are of course functions of z and temperature. *If* an "equivalent single-particle energy spectrum" ε_k exists, then we have

$$u_k=z\exp(-\beta\varepsilon_k)=\exp\beta(\mu-\varepsilon_k)\,. \qquad (2\cdot5)$$

Conversely, we may consider equation (2·5) as the *definition* of the energy spectrum ε_k. In that case, however, these energy levels ε_k may turn out to depend on temperature and (or) on the chemical potential.

As a first step let us assume $U_2=0$, i.e., the independent particle approximation. In that case it is easy to see that an acceptable operator is obtained by supplying a factor u_k for every "occupied" state k, and a factor 1 for all the unoccupied states. If we denote the independent-particle density matrix by $\mathcal{U}^0$, we therefore have

$$\mathcal{U}^0=\prod_k(u_k)^{n_k}=\prod_k(u_k)^{a_k^+a_k}. \qquad (2\cdot6)$$

The operator $\mathcal{U}^0$ is diagonal in the occupation-number representation. The operators a_k^+ and a_k are the usual creation and destruction operators, with the anti-commutation rules

$$\{a_k,\,a_{k'}^+\}_+=\delta_{kk'}\,. \qquad (2\cdot7)$$

To see the correspondence with (2·3), note that $N_2=0$ is now the only possibility; furthermore, we may omit the sum over permutations P provided that we restrict the sum over $k_1, k_2,\cdots, k_{N_1}$ to sets in which no index k occurs repeatedly. The factor $N_1!$ is then cancelled by the number of ways in which the indices $k_1,\cdots,k_{N_1}$ can be made to agree with a given pre-assigned "configuration".

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* The factor 2^{N_2} was omitted from equation (3·7) of reference 1) by mistake.

When pair correlations are included, i.e., when $U_2 \neq 0$, the appropriate formulation becomes less obvious. We shall first write down the final expression and then verify that it gives the trace (2·3) correctly. We decompose the correlation matrix U_2 as follows :

$$z^2\langle k_1 k_2|U_2|k_1' k_2'\rangle = \sum_\alpha w_\alpha(k_1, k_2)\, w_\alpha^*(k_1', k_2'). \tag{2·8}$$

Such a decomposition always exists, in fact there are infinitely many of them. In order to define the set of functions w_α uniquely, we would have to impose an additional requirement (e.g., that they are mutually orthogonal). However, it is neither necessary nor desirable to restrict the set w_α at this stage ; we shall find the freedom of choice of the w_α extremely useful later on.

We then use the functions $w_\alpha(k_1, k_2)$ to define second-quantized operators W_α by

$$W_\alpha = 2^{-1/2} \sum_{k_1 k_2} w_\alpha(k_1, k_2)\, a_{k_1} a_{k_2}. \tag{2·9}$$

If $w_\alpha(k_1, k_2)$ is not antisymmetric to start with, (2·9) selects its antisymmetric component.

Next, we define purely formal " labelling operators " A_α which operate on a separate Hilbert space, and obey the commutation rules

$$[A_\alpha, A_\beta^+]_- = \delta_{\alpha\beta}, \quad [A_\alpha, A_\beta]_- = [A_\alpha^+, A_\beta^+]_- = 0. \tag{2·10}$$

A state in the formal space is defined by the set of numbers N_α which are eigenvalues of th operators $A_\alpha^+ A_\alpha$. The vacuum state has $N_\alpha = 0$, all α. We denote by ω the projection operator onto this formal vacuum state. Finally, we define the operator Q by

$$Q = \sum W_\alpha A_\alpha^+. \tag{2·11}$$

Then the density matrix $\mathcal{U}$ of the system in the quasi-chemical equilibrium, or pair correlation, approximation is given by :

$$\mathcal{U} = \omega \exp(Q^+)\, \mathcal{V}\, \exp(Q)\, \omega. \tag{2·12}$$

In order to establish this Ansatz, we must show that the Trace of (2·12) equals (2·3). Rather than doing this for the general case at once, let us consider the special case in which U_2 is separable, i.e., the sum (2·8) can be reduced to one term. The extension to the general case is then an easy matter.

We therefore drop the index α. Expansion of the exponentials in power series gives

$$\mathcal{U} = \sum_{N,M=0}^{\infty} (N!M!)^{-1} \omega (AW^+)^N \mathcal{V} (A^+W)^M \omega. \tag{2·13}$$

We now observe that the commutation rules (2·10) imply the operator identity

$$\omega A^N (A^+)^M \omega = \delta_{N,M} N!\, \omega. \tag{2·14}$$

Substitution into (2·13) gives

$$\mathcal{U} = \sum_{N=0}^{\infty} (N!)^{-1} \omega (W^+)^N \mathcal{V} W^N = \sum_{N=0}^{\infty} (N!)^{-1} \omega \mathcal{W}_N, \tag{2·15}$$

where

$$\mathcal{W}_N = (W^+)^N \mathcal{V} W^N. \tag{2·16}$$

We are now to take the trace of (2·15) over the entire space, including the formal Hilbert space which we have introduced for the A-operators. However, the trace of the projection operator ω over this formal space is clearly unity; using the symbol TRACE to denote the full trace operation, and Trace to denote the trace over the physical Hilbert space only, we therefore get

$$\text{TRACE } \mathcal{U} = \sum_{N=0}^{\infty} (N!)^{-1} \text{ Trace } \mathcal{W}_N. \tag{2·17}$$

We note that the operator $\mathcal{W}_N$, (2·16), commutes with the number operator

$$\mathcal{N} = \sum_k a_k^+ a_k. \tag{2·18}$$

The factor W^N destroys $2N$ particles, $\mathcal{V}$ preserves the number of particles, and $(W^+)^N$ creates $2N$ particles.

Let us evaluate the trace of the operator $\mathcal{W}_N$. We write

$$\text{Trace } \mathcal{W}_N = \text{Trace } (W^+)^N \mathcal{V} W^N = \text{Trace } \mathcal{V} W^N (W^+)^N. \tag{2·19}$$

Let us consider a typical diagonal element of the operator on the extreme right in (2·19), for the configuration in which the states $k_1, k_2, \cdots, k_N$ are occupied, all others empty. The operator $\mathcal{V}$ acting on this state gives rise to the factor $v_{k_1} v_{k_2} \cdots v_{k_N}$. The other operator is equal to

$$\begin{aligned} W^N (W^+)^N = \sum_{m_1 \cdots m_{2N}} \sum_{m_1' \cdots m_{2N}'} & 2^{-N} w(m_1, m_2) w(m_3, m_4) \cdots w(m_{2N-1}, m_{2N}) \\ & \times w^*(m_1', m_2') \cdots w^*(m_{2N-1}', m_{2N}') \\ & \times a_{m_1} a_{m_2} \cdots a_{m_{2N}} a_{m_{2N}'}^+ \cdots a_{m_1'}^+. \end{aligned} \tag{2·20}$$

The diagonal element of this operator in the configuration $k_1 \cdots k_N$ vanishes if any of the m's agree with one of the k's; furthermore, the states $m_1' \cdots m_{2N}'$ must be a permutation of $m_1 \cdots m_{2N}$; in that case, the matrix element of the product of a- and a^+-operators is simply $(-1)^P$, where P is the permutation in question. When we combine the w and w^* factors back into $z^2 U_2$ matrices by using (2·8), we get the result

$$\begin{aligned} \text{Trace } \mathcal{W}_N = \sum_{s=0}^{\infty} \sum_{k_1 \cdots k_s}' \sum_{m_1 \cdots m_{2N}}' \sum_P' & (-1)^P 2^{-N} v_{k_1} v_{k_2} \cdots v_{k_s} \\ & \times \langle m_1 m_2 | z^2 U_2 | m_1' m_2' \rangle \cdots \langle m_{2N-1} m_{2N} | z^2 U_2 | m_{2N-1}' m_{2N}' \rangle. \end{aligned} \tag{2·21}$$

The sum over the k's is over configurations, whereas the sum over the m's is over the $2N$ indices independently. Let us now compare this result with (2·3), in particular with the term in (2·3) for which $N_1 = S$ and $N_2 = N$. There are the following differences:

1) The factor $(N_1!)^{-1}$ appears in (2·3) but not in (2·21); however, the sum over $k_1 \cdots k_N$ in (2·21) is over configurations, the sum over the k's in (2·3) is an independent sum; these two differences just cancel each other.

2) The permutations P in (2·21) permute only the $2N$ indices $m_1 \cdots m_{2N}$, whereas the permutations P in (2·3) permute all N_1+2N_2 indices. However, it was shown in reference 1) that the restriction to the smaller set of permutations is all right provided all the k's are different from each other, and different from all the m's (see formula (3·12) of reference 1)); these are just the restrictions indicated by the primes on the sums in (2·21).

3) The factor $(N_2!)^{-1}$ which appears in (2·3) is missing from (2·21). This is the only real difference.

Since this missing factor $(N!)^{-1}$ appears in (2·17), we conclude that the full trace of the operator $\mathcal{U}$, (2·12), is indeed equal to (2·3). Of course, the Ansatz (2·12) is not the only one which gives the correct trace; we are free to choose the off-diagonal elements of $\mathcal{U}$ quite arbitrarily. However, the choice of off-diagonal elements implied by (2·12) is clearly a "natural" choice, and we shall adhere to it henceforth.

The proof for the general case, in which the sum over α in (2·8) contains an arbitrary number of terms, goes through in an entirely analogous way; the operator identity replacing (2·14) is

$$\omega \prod_{\alpha} (A_\alpha)^{N_\alpha} (A_\alpha^+)^{M_\alpha} \omega = \prod_{\alpha} \delta_{N_\alpha M_\alpha} N_\alpha! \omega. \tag{2·22}$$

The labelling operators serve two functions in this formalism: (1) They ensure that there are exactly as many factors w_α as w_α^* in any surviving term; hence we can always combine the factors into $\langle k_1 k_2 | z^2 U_2 | k_1' k_2' \rangle$ by using (2·8), provided only that the combinatorial factors are right; (2) the labelling operators provide the necessary combinatorial factors for this purpose. The Bose commutation rules for the A-operators are essential for this, i.e., we need the $N_\alpha!$ in (2·22).

The N'th term in the expansion of the exponential $\exp(Q)$ in (2·12) thus "connects" only with the N'th term of the expansion of $\exp(Q^+)$, and their product gives the contribution of precisely N pair correlation, i.e., the term $N_2=N$ of (2·3), to the density matrix of the system. The $\mathcal{V}$ in (2·12) supplies a factor u_k for each occupied state k which is not already "used up" by the pair correlation terms.

The following two sections are devoted to re-deriving the main results of reference 1) from the form (2·12). An extension of the theory to include higher-order correlations is given in section 5. The self-consistency problem is discussed in section 6.

§ 3. Rederivation of the quasi-chemical equilibrium result

I. The leading term

Let us now show how we can use the formalism developed in section 2, to rederive in a simpler fashion the main results of reference 1). We wish to evaluate the trace:

$$\exp(-\beta\Omega) = \mathrm{Trace}\ \mathcal{U} = \mathrm{Trace}\,(\omega e^{Q^+} \mathcal{V}\ e^{Q} \omega). \tag{3·1}$$

We first use the fact that the operators in a trace can be permuted cyclically, to write this as

$$\exp(-\beta\Omega) = \mathrm{Trace}(\mathcal{V}\ e^{Q}\omega\, e^{Q^{+}}). \tag{3·2}$$

At this stage it helps to note that the projection operator onto the formal boson vacuum assures that there are as many factors W_α arising from the expansion of $\exp(Q)$ as there are factors W_α^{+} arising from the expansion of $\exp(Q^{+})$, separately for each value of α. We can achieve the same effect by placing operators ω to the left and to the right of the product of two exponentials, provided only that we interchange the roles of A_α and A_α^{+}. We define the operator P by

$$P = \sum_{\alpha} W_\alpha A_\alpha \tag{3·3}$$

i.e., P differs from Q, equation (2·11), by having a formal boson destruction operator instead of a formal boson creation operator. We then see that (3·2) can be rewritten as

$$\exp(-\beta\Omega) = \mathrm{Trace}(\mathcal{V}\ \omega\, e^{P}\, e^{P^{+}}\omega). \tag{3·4}$$

To prove the identity of (3·2) and (3·3), it is necessary merely to expand the exponentials and to use the identity (2·22).

Next, let us simplify (3·4). When we expand the exponentials and use (2·22), we see that a typical term contains the factor

$$a_{k_1}a_{k_2}\cdots a_{k_{2N}}a^{+}_{m_{2N}}a^{+}_{m_{2N-1}}\cdots a^{+}_{m_2}a^{+}_{m_1}$$

in which there are exactly as many a's as a^{+}'s. Furthermore, since the operator $\mathcal{V}$ is diagonal, the set $m_1, m_2, \cdots, m_{2N}$ must be a permutation of the set $k_1, k_2, \cdots, k_{2N}$, in order to give a non-zero diagonal element for the trace.

To see what happens, consider the simplest terms of this type, namely the traces:

$$\mathrm{Trace}(\mathcal{V}\, a_1 a_1^{+}) \quad \text{and} \quad \mathrm{Trace}(\mathcal{V}).$$

The Trace of $\mathcal{V}$, equation (2·6) is well-known to be:

$$\mathrm{Trace}\ \mathcal{V} = \prod_{k}(1+u_k). \tag{3·5}$$

When we evaluate the Trace of $\mathcal{V}\, a_1 a_1^{+}$, all indices $k \neq 1$ behave in the same way as before, so that we get the identity

$$\begin{aligned}\mathrm{Trace}(\mathcal{V}\, a_1 a_1^{+}) &= \Big[\prod_{k\neq 1}(1+u_k)\Big]\mathrm{Trace}(u_1^{a_1^{+}a_1}a_1 a_1^{+}) \\ &= [\mathrm{Trace}\ \mathcal{V}](1+u_1)^{-1}\,\mathrm{Trace}(u_1^{a_1^{+}a_1}a_1 a_1^{+}).\end{aligned} \tag{3·6}$$

In the last Trace in (3·6), the factor a_1 at the extreme right end assures that the contribution comes only from the vacuum state, i.e., the state with $n_1 = a_1^{+}a_1 = 0$. The factor $u_1^{n_1}$ then equals unity. Letting $|0\rangle$ stand for the vacuum state, we therefore have the identity

$$\mathrm{Trace}(\mathcal{V}\, a_1 a_1^{+}) = (\mathrm{Trace}\ \mathcal{V})\left\langle 0\left|\frac{a_1 a_1^{+}}{1+u_1}\right|0\right\rangle. \tag{3·7}$$

We can state this result more neatly by introducing the " quenched operators " $\tilde{a}_k$ through

$$\tilde{a}_k = (1+v_k)^{-1/2} a_k , \qquad \tilde{a}_k^+ = (1+v_k^+)^{-1/2} a_k^+ . \tag{3·8}$$

With this notation, we have

$$\text{Trace}(\mathcal{V} a_1 a_1^+) = (\text{Trace}\ \mathcal{V}) \langle 0|\tilde{a}_1 \tilde{a}_1^+|0\rangle . \tag{3·9}$$

At this stage it is easy to see that a similar relation holds for the general term arising from the expansion of the exponentials in (3·4) ; the product of $2Na$'s times $2Na^+$'s appears inside a vacuum expectation value, with all operators replaced by their " quenched " counterparts. The transformation from a to $\tilde{a}$, equation (3·8), is of course just the formal appearance of the reduction of interaction strengths due to the Pauli exclusion principle, the well-known statistical effect already discussed in reference 1).

When we now recombine terms, we get the following result for (3·4) :

$$\text{Trace}(\mathcal{U}) = \text{Trace}(\mathcal{V}\ \omega\, e^P\, e^{P^+} \omega) = \text{Trace}(\mathcal{V}) \langle 0|e^{\tilde{P}}\, e^{\tilde{P}^+}|0\rangle , \tag{3·10}$$

where $\tilde{P}$ is obtained from P, (3·3), by " quenching " ; explicitly,

$$\tilde{P} = \sum_\alpha \tilde{W}_\alpha A_\alpha = \frac{1}{\sqrt{2}} \sum_\alpha \sum_{k_1 k_2} \frac{w_\alpha(k_1, k_2)}{\sqrt{(1+v_{k_1})(1+v_{k_2})}}\, a_{k_1} a_{k_2} A_\alpha . \tag{3·11}$$

Equation (3·10) can be written in the form :

$$\exp(-\beta\Omega) = \exp(-\beta\Omega_F) \exp(-\beta\Omega_M) , \tag{3·12}$$

where Ω_F is the grand canonical potential for the ideal Fermi gas of " single " particles, and Ω_M is the grand canonical partition function for the " quasi-molecules " which result from the quenched pair correlations. We have thus recovered equations (3·18) and (3·19) of reference 1), and have obtained an alternative, but equivalent, expression for $\exp(-\beta\Omega_M)$ instead of (I.3.23), namely

$$\exp(-\beta\Omega_M) = \langle 0|e^{\tilde{P}}\, e^{\tilde{P}^+}|0\rangle . \tag{3·13}$$

Let us now show how (3·13) reduces, in first approximation, to the partition function of an ideal Bose gas. To do this, we first rewrite the Bose gas partition function in a different way. It is well known that the Bose partition function is

$$\exp(-\beta\Omega_B) = \prod_\alpha \frac{1}{1-\nu_\alpha} , \quad \text{(Ideal Bose Gas)} \tag{3·14}$$

where ν_α is related to the Bose particle energy level γ_α and to the chemical potential μ through

$$\nu_\alpha = \exp\beta(\mu - \gamma_\alpha) . \quad \text{(Ideal Bose Gas)} \tag{3·15}$$

Let us introduce Bose creation and annihilation operators A_α, A_α^+ in the usual way ; however we also introduce operators B_α, B_α^+ which operate in an independent space, but obey the same commutation rules :

$$[B_\alpha, B_\beta^+]_- = \delta_{\alpha\beta}, \tag{3·16}$$

$$[B_\alpha, B_\beta]_- = [B_\alpha^+, B_\beta^+]_- = 0. \tag{3·17}$$

We then assert that (3·15) is identically equal to the expression:

$$\exp(-\beta\Omega_B) = \langle 0|\exp(\sum_\alpha (v_\alpha)^{1/2} A_\alpha B_\alpha) \exp(\sum_\alpha (v_\alpha)^{1/2} A_\alpha^+ B_\alpha^+)|0\rangle. \tag{3·18}$$

The proof of this assertion is simple: First of all, operators involving different states α commute with each other, hence we get immediately:

$$\exp(-\beta\Omega_B) = \prod_\alpha \langle 0|\exp(v_\alpha^{1/2} A_\alpha B_\alpha) \exp(v_\alpha^{1/2} A_\alpha^+ B_\alpha^+)|0\rangle. \tag{3·19}$$

We now expand the two exponentials in power series, and make use of the fact that

$$\langle 0|(A_\alpha B_\alpha)^N (A_\alpha^+ B_\alpha^+) N'|0\rangle = \delta_{NN'} (N!)^2. \tag{3·20}$$

This gives:

$$\langle 0|\exp(v^{1/2} AB) \exp(v^{1/2} A^+ B^+)|0\rangle = \sum_{N,N'} \frac{v^{1/2\cdot(N+N')}}{N!N'!} \delta_{NN'} (N!)^2$$

$$= \sum_N v^N = \frac{1}{1-v}. \tag{3·21}$$

Combination of (3·19) and (3·21) proves the assertion (3·18).

We apply this result to obtain an approximate evaluation of (3·13). Let us now fix upon one particular decomposition of the quenched correlation matrix:*

$$\langle k_1 k_2|z^2 \tilde{U}_2|k_1' k_2'\rangle = \sum_\alpha \tilde{w}_\alpha(k_1, k_2) \tilde{w}_\alpha^*(k_1', k_2')$$

$$= \sum_\alpha \frac{w_\alpha(k_1, k_2) w_\alpha^*(k_1', k_2')}{V'(1+u_{k_1})(1+u_{k_2})(1+u_{k_1'})(1+u_{k_2'})}. \tag{3·22}$$

We fix the decomposition by requiring the functions $\tilde{w}_\alpha(k_1, k_2)$ to form an orthogonal set; these functions are then related to the eigenfunctions of the matrix $z^2 U_2$ through

$$\tilde{w}_\alpha(k_1, k_2) = v_\alpha^{1/2} \phi_\alpha(k_1, k_2), \tag{3·23}$$

where v_α and $\phi_\alpha(k_1, k_2)$ are the eigenvalues and normalized eigenfunctions of $z^2 U_2$, respectively.

We then introduce "physical boson" annihilation operators b_α by

$$b_\alpha = \frac{1}{V'} \sum_{k_1, k_2} \phi_\alpha(k_1, k_2) a_{k_1} a_{k_2}. \tag{3·24}$$

In terms of these operators, equation (3·11) becomes:

$$\tilde{P} = \sum_\alpha v_\alpha^{1/2} b_\alpha A_\alpha. \tag{3·25}$$

* It is at this point that we make essential use of the freedom of choice of the decomposition of the original operator $z^2 U_2$, equation (2·8). If we had insisted, at that earlier stage, that the functions w_α are to be orthogonal to each other, then the quenched functions w_α would *not* form an orthogonal set.

When we insert (3·25) into (3·13), we obtain precisely the form (3·18) for the partition function of the ideal Bose gas, with one exception: the physical boson operators b_α, b_α^+ obey the commutation rule (3·17), but they do not obey the rule (3·16). Rather, the commutator is

$$[b_\alpha, b_\beta^+]_- = \delta_{\alpha\beta} + \sum_{k_1,k_2,k'} \phi_\alpha(k_1, k')\phi_\beta^*(k', k_2)\, a_{k_2}^+ a_{k_1}. \qquad (3\cdot26)$$

The additional term by which (3·26) differs from (3·17a) is an expression of the fact that the "quasi-molecules" described by the eigenfunctions ϕ_α are not really independent Bose-Einstein particles; rather they are complex structures, whose constituent particles obey Fermi-Dirac statistics.

Note that the requirement that the $\tilde{w}_\alpha$ form an orthogonal set is essential to obtain the leading term $\delta_{\alpha\beta}$ of (3·26). Note also that b_α, equation (3·24), is the operator one would naturally write down in second-quantization formalism as the wave-operator associated with the first-quantized wave function ϕ_α.

To the extent that the overlap integrals which appear in (3·26) can be ignored, the form (3·13) together with (3·25) is equivalent to an ideal Bose gas partition function, (3·18). Of course, equation (3·15) fails completely for this system; or, if we insist on *defining* "energy levels" γ_α of our "quasi-molecules" by the use of equation (3·15), then the γ_α depend in some complicated way on temperature and on the chemical potential μ.

§ 4. Re-derivation of the quasi-chemical equilibrium result
II. The higher terms

The difficulty in evaluating the "quasi-molecule" partition function, (3·13), arises because of the altered commutation rules, (3·26) instead of (3·16): the "physical boson" creation and destruction operators b_α^-, b_β obey commutation rules which differ slightly from the "ideal boson" operators B_α^+, B_β.

It is easier to work with operators which obey the "ideal" commutation rules, (3·16), and to transform the additional term in the commutation rules for the "physical boson" operators into an equivalent, additional term for the *operator* whose vacuum expectation value we desire. This can be achieved by means of a method developed by Dyson[2] in connection with the theory of spin waves.

Let us first define a "physical boson" configuration $|n_1, n_2, \cdots\rangle = |a\rangle$ as the state

$$|a\rangle = |n_1 n_2 n_3 \cdots\rangle = \prod_\alpha \frac{(b_\alpha^+)^{n_\alpha}}{\sqrt{n_\alpha!}}|0\rangle, \qquad (4\cdot1)$$

where $|0\rangle$ is the vacuum state, and the b_α^+ operators are the Hermitian adjoints of the b_α, equation (3·24). Since all b_α^+ commute with each other, the order of factors in the product is irrelevant.

The set of states $|a\rangle$ is complete, but not orthogonal; as an example, consider two "two-molecule" configurations, say $|a\rangle = |1_\alpha 1_\beta\rangle$ and $|b\rangle = |1_\gamma 1_\delta\rangle$. Their overlap integral is

$$\langle 1_\alpha 1_\beta | 1_\gamma 1_\delta \rangle = \langle 0 | b_\alpha b_\beta b_\gamma^+ b_\delta^+ | 0 \rangle = \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} - C_{\gamma\delta}^{\alpha\beta}, \tag{4·2}$$

where

$$C_{\gamma\delta}^{\alpha\beta} = \sum_{k_1 k_2 k_3 k_4} 2^2 \phi_\alpha(k_1, k_2)\phi_\beta(k_3, k_4)\phi_\gamma^*(k_2, k_3)\phi_\delta^*(k_4, k_1). \tag{4·3}$$

Note that the k's which appear as arguments of the complex conjugate wave functions in this formula are the result of performing the cyclic permutation (1, 2, 3, 4) on the k's which appear as arguments of the unstarred wave functions. In the reduction of (4·2) to (4·3) we have made use of the fact that all the ϕ's are antisymmetric functions under interchange of the two k's. Using this, it is also seen easily that

$$C_{\gamma\delta}^{\alpha\beta} = C_{\gamma\delta}^{\beta\alpha} = C_{\delta\gamma}^{\alpha\beta} = (C_{\alpha\beta}^{\gamma\delta})^*. \tag{4·4}$$

Although "physical boson" states $|a\rangle$ are neither orthogonal nor normalized in general, there is one exception to this rule: the vacuum state $|0\rangle$ is not only normalized, but is also orthogonal to all other physical boson states:

$$\langle a | 0 \rangle = \delta_{a,0}. \tag{4·5}$$

Before we go on, let us note that the quantities C of equation (4·3) enter into the commutation relations between physical boson operators. To be precise, the double commutator between one b and two b^+'s is given by

$$[[b_\alpha, b_\gamma^+]b_\delta^+] = -\sum_\beta C_{\gamma\delta}^{\alpha\beta} b_\beta^+. \tag{4·6}$$

Let us now, following Dyson, consider a correspondence between the physical boson states (4·1) and operators T acting on them, and "ideal boson states" $|a)$ and operators $\boldsymbol{T}$ acting on them. We use ideal boson operators B_α, B_α^+ obeying the ordinary Bose commutation rules (3·16) to construct the ideal boson state $|a)$ as

$$|a) = |n_1 n_2 n_3 \cdots) = \prod_\alpha \frac{(B_\alpha^+)^{n_\alpha}}{\sqrt{n_\alpha!}} |0), \tag{4·7}$$

where $|0)$ is the ideal boson vacuum state. It is easily seen that the states (4·7) are orthonormal:

$$(a|a') = \delta_{a,a'} \equiv \prod_\alpha \delta_{n_\alpha n_{\alpha'}}. \tag{4·8}$$

We then set up a linear mapping $T \to \boldsymbol{T}$ of operators in the two spaces: let coefficients $T_{aa'}$ be *defined* by

$$T|a'\rangle = \sum_a T_{aa'}|a\rangle. \quad \text{(Definition of } T_{aa'}) \tag{4·9}$$

Because of the lack of orthonormality of the set $|a\rangle$, $T_{aa'}$ is *not* equal to the matrix element $\langle a|T|a'\rangle$. The corresponding operator $\boldsymbol{T}$ in the ideal boson space is *defined* by

$$\boldsymbol{T}|a') = \sum_a T_{aa'}|a). \quad \text{(Definition of } \boldsymbol{T}) \tag{4·10}$$

Since the ideal boson states $|a)$ are properly orthonormal, it follows that

$$T_{aa'} = (a|\boldsymbol{T}|a'). \tag{4·11}$$

The mapping $T \leftrightarrow \boldsymbol{T}$ is single-valued and linear, i.e., $T_1 + T_2$ maps into $\boldsymbol{T}_1 + \boldsymbol{T}_2$, $T_1 T_2$ maps into $\boldsymbol{T}_1 \boldsymbol{T}_2$, the zero operators map into each other, and so do the unit operators. However, the mapping operation does *not* preserve scalar products, and therefore also fails to preserve the correspondence between an operator T and its adjoint T^+. For example, it is obvious from the definitions given so far that b_α^+ maps into B_α^+, i.e.

$$b_\alpha^+ \leftrightarrow \boldsymbol{b}_\alpha^+ = B_\alpha^+. \tag{4·12}$$

On the other hand, b_α definitely does *not* map into B_α. Indeed, our next task will be the construction of $\boldsymbol{b}_\alpha$.

Let us define the operator $\boldsymbol{O}^+(a)$ as the ideal boson operator which generates the state $|a)$ from the vacuum state $|0)$, i.e., according to (4·7),

$$\boldsymbol{O}^+(a) = \prod_\alpha \frac{(B_\alpha^+)^{n_\alpha}}{\sqrt{n_\alpha!}}. \tag{4·13}$$

Let us also define, for an arbitrary operator T in the original space, a sequence of coefficients by multiple commutation with the b^+ operators:

$$T|0\rangle = \sum_a T_{a0}|a\rangle, \quad \text{(Definition of } T_{a0}) \tag{4·14a}$$

$$[T, b_\alpha^+]|0\rangle = \sum_a T_{a0}^\alpha |a\rangle, \quad \text{(Definition of } T_{a0}^\alpha) \tag{4·14b}$$

$$[[T, b_\alpha^+]b_\beta^+]|0\rangle = \sum_a T_{a0}^{\alpha\beta}|a\rangle, \quad \text{(Definition of } T_{a0}^{\alpha\beta}) \tag{4·14c}$$

and so on. It is sometimes true that the sequence of coefficients defined by (4·14) is easier to obtain explicitly than the general coefficient $T_{aa'}$ defined by equation (4·9). If these coefficients are known, then we can obtain the corresponding ideal operator $\boldsymbol{T}$ by the following:

Lemma: The ideal operator $\boldsymbol{T}$ corresponding to a physical operator T with coefficients given by (4·14) is:

$$\boldsymbol{T} = \sum_a \boldsymbol{O}^+(a)\left[T_{a0} + \frac{1}{1!}\sum_\alpha T_{a0}^\alpha B_\alpha + \frac{1}{2!}\sum_{\alpha\beta} T_{a0}^{\alpha\beta} B_\alpha B_\beta + \cdots\right]. \tag{4·15}$$

Proof: Since the mapping $T \leftrightarrow \boldsymbol{T}$ preserves sums and products, it preserves commutators. Since the mapping also maps b_α onto B_α^+, we can get a check on the correctness of (4·15) by evaluating the series of matrix elements, in the *ideal* space, which are involved in the definitions (4·14) in the physical boson space. That is, we wish to check the relations

$$(a|\boldsymbol{T}|0) = T_{a0} \tag{4·16a}$$

$$(a|[T, B_\alpha^+]|0) = T_{a0}^\alpha, \tag{4·16b}$$

$$(a|[[T, B_\alpha^+], B_\beta^+]|0) = T_{a0}^{\alpha\beta}, \tag{4·16c}$$

...

When we insert the form (4·15) for $\boldsymbol{T}$ in (4·16), the operators $\boldsymbol{O}^{-}(a)$ always commute through, and we use the Bose commutation rules for the B's and B^{+}'s. When we do so, the relations (4·16) are easily seen to be identities.

To complete the proof, it remains merely to observe that these checks are also exhaustive, in that all matrix elements $(a|\boldsymbol{T}|b)$ are eventually involved. For example, the double commutator which appears in (4·16c) involves, among other terms, the term: $\boldsymbol{T}B_{\alpha}^{-}B_{\beta}^{+}$. When this operates on the vacuum state $|0\rangle$, it is equivalent to $\boldsymbol{T}$ itself operating on the two-boson state $|1_{\alpha}1_{\beta}\rangle$, and hence the check (4·16c) provides a check on matrix elements $(a|\boldsymbol{T}|b)$ with $|b\rangle$ any two-boson state.

We are now in a position to construct the ideal operator $\boldsymbol{b}_{\alpha}$ corresponding to the physical boson operator b_{α}. From the definition of b_{α}, (3·24), and relations (3·26) and (4·6), we get

$$
\begin{aligned}
&b_{\alpha}|0\rangle=0 \quad \text{so that} \quad (b_{\alpha})_{a0}=0.\\
&[b_{\alpha},\, b_{\beta}^{+}]|0\rangle=\delta_{\alpha\beta}|0\rangle \quad \text{so that} \quad (b_{\alpha})_{a0}^{\beta}=\delta_{a\beta} \quad \text{for} \quad |a\rangle=|0\rangle\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =0 \quad \text{otherwise.}\\
&[[b_{\alpha},\, b_{\gamma}^{+}],\, b_{\delta}^{+}]|0\rangle=-\sum_{\beta} C_{\gamma\delta}^{\alpha\beta}|1_{\beta}\rangle \quad \text{so that} \quad (b_{\alpha})_{a0}^{\gamma\delta}=-C_{\gamma\delta}^{\alpha\beta} \quad \text{for} \quad |a\rangle=|1_{\beta}\rangle\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad =0 \quad \text{otherwise.} \qquad (4\cdot17)
\end{aligned}
$$

Because of (4·6) all higher commutators vanish identically, and thus the series (4·15) terminates. In fact, we have:

$$
\boldsymbol{b}_{\alpha}=B_{\alpha}-\tfrac{1}{2}\sum_{\beta\gamma\delta} C_{\gamma\delta}^{\alpha\beta}\, B_{\beta}^{+}B_{\gamma}B_{\delta}\,. \qquad (4\cdot18)
$$

Since products map into products and sums into sums, the exponential function of an operator maps into the exponential function of the transformed operator. Thus we get the following mapping

$$
\exp(\boldsymbol{P})\exp(\boldsymbol{P}^{+})=\exp\Big[\sum_{\alpha} r_{\alpha}^{1/2}A_{\alpha}\Big(B_{\alpha}-\tfrac{1}{2}\sum_{\beta\gamma\delta} C_{\gamma\delta}^{\alpha\beta}\, B_{\beta}B_{\gamma}B_{\delta}\Big)\Big]\exp\Big[\sum_{\alpha} r_{\alpha}^{1/2}A_{\alpha}B_{\alpha}^{+}\Big]\,. \qquad (4\cdot19)
$$

Furthermore, although in general the coefficient $T_{aa'}$ in the expansion (4·9) is not equal to the matrix element $\langle a|T|a'\rangle$, the vacuum expectation value $\langle 0|T|0\rangle$ is an exception to this rule. When we premultiply equation (4·14a) by $\langle 0|$ and use relation (4·5), we get immediately

$$
T_{00}=\langle 0|T|0\rangle
$$

and hence, from the general relation (4·11),

$$
\langle 0|T|0\rangle=(0|\boldsymbol{T}|0)\,. \qquad (4\cdot20)
$$

When we use expression (4·19) for $\boldsymbol{T}$ in (4·20), we obtain $\exp(-\beta\Omega_{M})$, equation (3·13), as the vacuum expectation value of the operator (4·19), with the A and B operators obeying ordinary Bose commutation rules. Thus we have achieved our first purpose, to convert an expression involving operators with "not-quite-Bose" commutation

rules into an equivalent expression involving only ordinary Bose operators. Let us introduce some notation :

$$R=\sum_{\alpha} v_{\alpha}^{1/2} A_{\alpha} B_{\alpha} , \tag{4·21a}$$

$$R^{+}=\sum_{\alpha} v_{\alpha}^{1/2} A_{\alpha}^{+} B_{\alpha}^{+} , \tag{4·21b}$$

$$S=-\tfrac{1}{2} \sum_{\alpha\beta\gamma\delta} C_{\gamma\delta}^{\alpha\beta} v_{\alpha}^{1/2} A_{\alpha} B_{\beta}^{+} B_{\gamma} B_{\delta} . \tag{4·21c}$$

Then we have the identity :

$$\exp(-\beta\Omega_M) = (0|e^{R+S}e^{R^+}|0) . \tag{4·22}$$

In order to make an expansion, let us multiply the operator S by a c-number parameter λ, and expand Ω_M as a power series in λ ; finally we set $\lambda=1$. We get

$$\exp(-\beta\Omega_M(\lambda)) = (0|e^{R+\lambda S}e^{R^+}|0) \tag{4·23}$$

with

$$\Omega_M(\lambda) = \Omega_1+\lambda\Omega_2+\lambda^2\Omega_3+\cdots \tag{4·24}$$

and

$$\Omega_s = \frac{1}{(s-1)!}\left[\frac{\partial^{s-1}\Omega_M(\lambda)}{\partial\lambda^{s-1}}\right]_{\lambda=0} = \frac{-1}{\beta(s-1)!}\left[\frac{\partial^{s-1}}{\partial\lambda^{s-1}} ln(0|e^{R+\lambda S}e^{R^+}|0)\right]_{\lambda=0} . \tag{4·25}$$

For reasons which will become apparent later, we shall restrict ourselves to the evaluation of the first two terms of the series (4·24). The leading term Ω_1 is of course just the Bose result obtained in section 3. For the next term, Ω_2, we use the relation :

$$\left[\frac{d}{d\lambda}\exp(R+\lambda S)\right]_{\lambda=0} = \{\exp(R+\lambda S)\int_0^1 dt \exp[-t(R+\lambda S)]S\exp[t(R+\lambda S)]\}_{\lambda=0}$$

$$= e^{R}\int_0^1 dt\, e^{-tR}Se^{tR}. \tag{4·26}$$

The integrand in the last expression in (4·26) is evaluated most easily by using the Bose commutation rules together with the general identity :

$$e^{-tR}Se^{tR} = S-\int_0^t dt'\, e^{-t'R}[R, S]e^{t'R}. \tag{4·27}$$

The commutator of R, (4·21a), and S, (4·21c), commutes with R so that the integration becomes trivial. We then get

$$\int_0^1 dt\, e^{-tR}Se^{tR} = -\tfrac{1}{2}\sum_{\alpha\beta\gamma\delta} C_{\gamma\delta}^{\alpha\beta} v_{\alpha}^{1/2} A_{\alpha}(B_{\beta}^{+}-\tfrac{1}{2}v_{\beta}^{1/2}A_{\beta})B_{\gamma}B_{\delta} \equiv T. \tag{4·28}$$

When we insert (4·26) and (4·28) into (4·25) for the case $s=2$, we find

$$-\beta\Omega_2 = \frac{(0|e^{R}Te^{R^+}|0)}{(0|e^{R}e^{R^+}|0)} . \tag{4·29}$$

The ratio (4·29) splits into a product of independent ratios, one for each state α. Those indices which do not occur in a particular term of the sum (4·28) make a unit contribution to the overall ratio; for those indices which do occur, the evaluation is most easily done by the trivial method of expanding the exponential in a power series, using the commutation rules, and collecting terms at the end. The result is

$$-\beta\Omega_2=-\tfrac{1}{4}\sum_{\alpha\beta\gamma\delta} C_{\gamma\delta}^{\alpha\beta}(\delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma})\frac{v_\alpha}{1-v_\alpha}\frac{v_\beta}{1-v_\beta}.$$

Using the symmetry relations (4·4), this becomes

$$-\beta\Omega_2=-\tfrac{1}{4}\sum_{\alpha,\beta} C_{\alpha\beta}^{\alpha\beta}\frac{v_\alpha}{1-v_\alpha}\frac{v_\beta}{1-v_\beta}. \tag{4·30}$$

Let us now compare this result with the corresponding expression, formula (II. 18), of reference 1). It is easily seen that the quantity $\Psi_2(\alpha,\beta)$ appearing there is equal to $-\frac{1}{2}C_{\alpha\beta}^{\alpha\beta}$ in our notation; of course z^2u_α equals our v_α. It is apparent that the expressions for Ω_2 differ significantly. In tracing through the earlier calculation, we have found a mistake in the combinatorial factor of equation (II. 5). The mistake is rectified by replacing

$$2^{l-s}(l-s)! \quad \text{by} \quad 2^{l-s}\frac{l!}{s!}.$$

The effect of this replacement is to change expression (II. 16) for $f_s(x)$ into the much simpler form

$$f_s(x)=\sum_{l=s}^{\infty}{}' x^l=x^s\ (1-x).$$

When this is substituted into (II. 15) and appropriate reductions are carried out, the final formula is extremely neat, namely

$$-\beta\Omega_M^{(s)}=\sum_{\alpha_1\cdots\alpha_s}\Psi_s(a_1,\cdots,a_s)\prod_{i=1}^{s}\frac{v_{\alpha_i}}{1-v_{\alpha_i}}. \tag{4·31}$$

(4·30) is the special case of (4·31) for $s=2$. We remark, incidentally, that the higher terms Ω_s of our expansion (4·24) do *not* agree, term by term, with the higher terms $\Omega^{(s)}$ of (4·31). For example, our Ω_3 contains not only Ω^3 but also part of Ω^4. However, as we shall now see, this makes no difference whatsoever, since neither expansion can be used near the condensation point!

Condensation occurs when one of the $v_\alpha=z^2u_\alpha$ approaches unity. In order for an expansion of the type considered here to be useful near condensation, it is necessary that the higher terms Ω_s or $\Omega^{(s)}$ have singularities at this point no stronger than the singularity of the leading term Ω_1, which is of course a logarithmic singularity. However, it is obvious that this condition fails to hold. For example, Ω_2 gives rise to a quadratic singularity near condensation, arising from the term in (4·30) with $\alpha=\beta=\alpha_0$, where v_{α_0} is that particular eigenvalue of the pair correlation operator which approaches unity first.

Following the discussion at the end of Appendix II of reference 1), we are forced to the conclusion that the present evaluation of the formalism does not allow us to discuss the nature of the condensation phenomenon, since the series becomes useless near the condensation point. The new evaluation of the formalism necessary for this will be given in two subsequent papers. Suffice it to say here that the essential features are unchanged. The condensation is still an ordinary Bose-Einstein condensation of the "quasi-molecules" which appear as eigenfunctions of the quenched correlation matrix. The main difference is that the condensation occurs at a different value of the chemical activity z. This implies immediately that any power series expansion around the ideal Bose result must fail near condensation, since the *location* of the singularity of Ω_M, as a function of z, has shifted. This is true no matter how small the amount of the shift may be.

§ 5. Extension of the formalism to higher-order correlations

In the quasi-chemical equilibrium approximation, we take into account dynamical pair correlations (and statistical correlations of all orders). We shall now show how the labelling operator formalism can be extended to cover the general case of dynamical correlations of all orders. As before, we use the grand canonical density matrix; the trace of that matrix appears naturally as a sum of terms, each term giving the contribution of a combination of N_1 "single particles", N_2 "pair correlations", N_3 "triplet correlations", N_4 "quadruplet correlations", and so on. The formula for the trace is a straight-forward extension of equation (2·3) for the quasi-chemical equilibrium approximation:

$$\begin{aligned}\text{Trace } \mathcal{U} = \sum_{N_1, N_2, N_3 \cdots} \sum_{k_1, \cdots, k_{N_1}} \sum_{l_1, \cdots, l_{2N_2}} \sum_{m_1, \cdots, m_{3N_3}} \cdots \sum_{P} \\ \frac{(-1)^P}{(1!)^{N_1}(2!)^{N_2}(3!)^{N_3}\cdots N_1!\,N_2!\,N_3!\cdots} \langle k_1|zU_1|k_1^P\rangle\langle k_2|zU_1|k_2^P\rangle\cdots\langle k_{N_1}|zU_1|k_{N_1}^P\rangle \\ \times\langle l_1 l_2|z^2U_2|l_1^P l_2^P\rangle\cdots\langle l_{2N_2-1}l_{2N_2}|z^2U_2|l_{2N_2-1}^P l_{2N_2}^P\rangle \\ \times\langle m_1 m_2 m_3|z^3U_3|m_1^P m_2^P m_3^P\rangle\cdots\langle m_{3N_3-2}m_{3N_3-1}m_{3N_3}|z^3U_3|m_{3N_3-2}^P m_{3N_3-1}^P m_{3N_3}^P\rangle \\ \cdots \qquad (5\cdot1)\end{aligned}$$

Here U_2 is the Ursell matrix for pair correlations, U_3 the Ursell matrix for triplet correlations, and so on. The permutations P transform the set of $N_1+2N_2+3N_3+\cdots$ indices $k_1,\cdots, k_{N_1}$, $l_1,\cdots l_{2N_2}$, $m_1,\cdots, m_{3N_3},\cdots$ into the permuted indices k_1^P etc. Just as before, we would like to construct an operator $\mathcal{U}$ in second-quantization form, such that the trace of $\mathcal{U}$ is given by (5·1).

Reasoning by analogy with the pair correlation case, we expect that it will prove useful to decompose the higher-order correlation matrices into sums of products; in general, for an s-particle correlation:

$$\langle k_1, k_2, \cdots, k_s|z^sU_s|k_1', k_2', \cdots, k_s'\rangle = \sum_\alpha w_\alpha^{(s)}(k_1, \cdots, k_s)\, w_\alpha^{(s)*}(k_1', \cdots, k_s'). \qquad (5\cdot2)$$

Equation (2·8) for the pair correlations is the special case $s=2$ of (5·2). We then define second quantization operators $W_\alpha^{(s)}$ associated with each component $w_\alpha^{(s)}$ through:

$$W_\alpha^{(s)}=\frac{1}{\sqrt{s!}}\sum_{k_1,\cdots,k_s} w_\alpha^{(s)}(k_1,\cdots,k_s)\,a_{k_1}a_{k_2}\cdots a_{k_s}. \tag{5·3}$$

If the w_α is already a completely anti-symmetric function of its s arguments, then there are $s!$ equal terms in (5·3) for each set of s different states $k_1,\cdots,k_s$. If the w_α are not all antisymmetric, the formula (5·3) automatically selects only the antisymmetric components of the w_α. Equation (2·9) is the special case $s=2$ of (5·3).

When we now attempt to place factors W and W^+ on either side of the single-particle operator $\mathcal{V}$, the need for labelling operators of some kind becomes immediately apparent: we must finally be able to combine terms so as to return to expressions involving only the matrices ζU_ν, independently of their (non-unique) decompositions (5·2). Hence there must be exactly as many factors $w_\alpha^{(s)}$ on one side as there are factors $w_\alpha^{(s)*}$ on the other side of $\mathcal{V}$, separately for each s and each α. We must therefore introduce labelling operators $A_\alpha^{(s)}$. Since these are purely formal operators, acting in a purely formal space, we are free to choose their commutation rules.

We construct the operator $Q^{(s)}$ in analogy to (2·11), i.e.,

$$Q^{(s)}=\sum_\alpha W_\alpha^{(s)}A_\alpha^{(s)+}. \tag{5·4}$$

We would like to impose the condition that the different operators which appear in this sum commute with each other. Only then can we make efficient use of the exponential function of $Q^{(s)}$. However, it is obvious that, for odd values of s, the terms of (5·4) commute with each other only if the $A_\alpha^{(s)}$ obey Fermi (anti-)commutation rules. That is, we are led to the commutation rules:

$$[A_\alpha^{(2s)},\ A_\beta^{(2t)+}]_-=\delta_{st}\delta_{\alpha\beta}, \tag{5·5a}$$

$$\{A_\alpha^{(2s+1)},\ A_\beta^{(2t+1)+}\}_+=\delta_{st}\delta_{\alpha\beta}, \tag{5·5b}$$

$$[A_\alpha^{(s)},\ A_\beta^{(t)}]_-=[A_\alpha^{(s)+},\ A_\beta^{(t)+}]_-=0 \quad \text{if at least one of } s,\ t, \text{ is even}, \tag{5·5c}$$

$$\{A_\alpha^{(2s+1)},\ A_\beta^{(2t+1)}\}_+=\{A_\alpha^{(2s+1)+},\ A_\beta^{(2t+1)+}\}_+=0 \text{ all } s,\ t, \tag{5·5d}$$

$$[A_\alpha^{(2s)},\ A_\beta^{(2t+1)+}]_-=0 \text{ all } s,\ t. \tag{5·5e}$$

With these rules, all the operators $W_\alpha^{(s)}A_\alpha^{(s)+}$ commute with each other.

For *odd* s, the rules (5·5b) and (5·5d) imply that a given index α can occur *at most once* in the decomposition of a product of N_ν factors ζU_ν of the type appearing in (5·1). This is all right because, for all odd values of s, the following identity holds:

$$(W_\alpha^{(s)})^2=(W_\alpha^{(s)+})^2=0 \quad \text{for all } \textit{odd } s. \tag{5·6}$$

With this operator identity, it is clear that terms in which an index α is repeated never occur, no matter how the labelling is accomplished. Thus we can use Fermi rules for the $A_\alpha^{(s)}$ without killing off needed terms.

To prove (5·6), consider, for example, the case $s=3$. The square of $W^{(3)}$ (omitting the index α) is, according to (5·3),

$$(W^{(3)})^2=(3!)^{-1}\sum_{m_1\cdots m_6} w^{(3)}(m_1, m_2, m_3)\, w^{(3)}(m_4, m_5, m_6)\, a^{(3)}_{m_1}\cdots a^{(3)}_{m_6}\cdot \qquad (5\cdot 7)$$

Let us focus our attention on two ordered sets of three indices, say k_1, k_2, k_3 and l_1, l_2, l_3. These sets appear twice, namely:

$$m_1=k_1,\ m_2=k_2,\ m_3=k_3,\ m_4=l_1,\cdots,\ m_6=l_3 \text{ and } m_1=l_1,\cdots,\ m_6=k_3\,.$$

The numerical coefficient $w^{(3)}w^{(3)}$ is the same, and the operators $a\cdots a$ just cancel to give zero, because of the anti-commutation rules. This cancellation occurs for all odd values of s, thereby completing the proof.

As a consequence of this fact, the special case of "one quantum state correlations", i.e. one term only in the sum over α in (5·2), is of no interest whatever for odd s. A correlation matrix of rank 1 for odd-order correlations has no influence on the statistical mechanics of the system; for it can appear as at most one factor in a product of infinitely many factors, such as (5·1). By the same reasoning, odd-order Ursell matrices of any *finite* rank (the sum over α is a finite sum) can be ignored altogether.

For even s, on the other hand, there is no restriction on indices α repeating, and in fact the Bose-type condensation which may occur in this formalism just means that one index α occurs a macroscopic number of times, below a certain critical temperature.

Of course, all this represents merely the formal appearance of the well-known fact that molecules composed of an even number of Fermions obey Bose statistics (in some reasonable approximation), molecules composed of an odd number of Fermions obey Fermi statistics. It is pleasant to find, however, that the expected connection appears so naturally and unavoidably in the formalism developed here.

From here on, we just proceed as before. We define the operator Q by

$$Q=\sum_{s=2}^{s_{\max}} Q^{(s)}, \qquad (5\cdot 8)$$

where $s_{\max}$ is the highest order dynamical correlation retained in the description of the system. As a result of the commutation rules chosen, all terms of the sum (5·8) commute with each other. The density matrix of the system is then given by:

$$\mathcal{U}=\omega \exp(Q^+)\, \mathcal{V}\, \exp(Q)\, \omega, \qquad (5\cdot 9)$$

where $\mathcal{V}$ is defined by (2·6) and ω is the projection operator for the vacuum state of all the formal operators A^+A.

We shall omit the detailed proof that the statistical factors $N!$ work out all right when we take the trace of (5·9) and compare with (5·1). There is no difficulty with this proof, once the identity (5·6) is known to exist.

In closing this section, we make two remarks:

1) If the sum over s in (5·8) is extended over *all* $s\geq 2$, we obtain a complete expansion for the density matrix $\mathcal{U}$. Under certain special conditions (for example, the

hard sphere gas studied by Yang and Lee[7]) such a complete expansion may prove useful. It supplies an algebraic method of handling the "graphs" which appear in such a formalism. Our own interest, however, lies in approximations in which the sum over s is restricted to a small number of terms, and the influence of higher order terms is taken into account in some self-consistent way. After all, the terms of importance are essentially of two types: small s, including of course the independent particle terms; and very large s, which give "collective motions". If collective motions are not important for the problem at hand a finite sum over s is sufficient; if the collective motions are important, then they should be handled as such, by some approximation appropriate for that purpose, rather than by the formal device of extending the sum over s in (5·8) to extremely high s.

2) The "quenching factors" by which the Pauli exclusion principle inhibits the effects of higher order correlations appear naturally in the extended formalism. Indeed, by precisely the same arguments as before (see section 3), we arrive at the factored expressson:

$$\mathrm{Trace}\,\mathcal{U}=\mathrm{Trace}\,\mathcal{V}\langle 0|\exp(\tilde{P})\exp(\tilde{P}^{+})|0\rangle, \tag{5·10}$$

where

$$\tilde{P}=\sum_{s=2}^{s_{\max}}\tilde{P}_s=\sum_{s=2}^{s_{\max}}\sum_{\alpha}W_{\alpha}^{(s)}A_{\alpha}^{(s)} \tag{5·11}$$

and

$$W_{\alpha}^{(s)}=\frac{1}{\sqrt{s!}}\sum_{k_1\cdots k_s}\frac{w_{\alpha}^{(s)}(k_1,\cdots,k_s)}{\sqrt{(1+u_{k_1})(1+u_{k_2})\cdots(1+u_{k_s})}}a_{k_1}\cdots a_{k_s}. \tag{5·12}$$

However, whereas for $s_{\max}=2$, this reduction is already a major step towards the explicit evaluation of the trace, for the general case a lot remains to be done. The vacuum expectation value which appears in (5·10) contains the contributions of *all* the dynamical correlations, and is not separated into a product of factors, one factor for pair correlations, one for triplet correlations, and so on.

§ 6. Conditions on the Ursell matrices

So far, we have taken the Ursell matrices U_1, U_2, etc. as given quantities; we have shown (in sections 2 and 6) how the statistical operator $\mathcal{U}$ can be constructed from them, and (in sections 3 and 4) how the trace of $\mathcal{U}$ can be evaluated in terms of properties of these Ursell matrices.

However, we also need some way to find the matrices U_s, starting from fundamentals. In general, our information about a thermodynamic system consists of microscopic information, namely the Hamiltonian H of the system, and thermodynamic information, which specifies a unique thermodynamic state of the system. This latter information can be given in many ways; for our present purpose we shall assume that we are given the internal energy of the system, i.e.,

$$E = \text{Trace}\,(H\,\mathcal{U}) \tag{6·1}$$

and the number of particles

$$N = \text{Trace}\,(\mathcal{N}\,\mathcal{U}) \tag{6·2}$$

where $\mathcal{N}$ is the operator (2·18). In these equations, $\mathcal{U}$ is the *normalized* statistical operator, i.e.,

$$\text{Trace}\,(\mathcal{U}) = 1. \tag{6·3}$$

We shall consider systems at constant volume V throughout.

Since the Trace of expression (5·9) is not equal to unity in general, we must alter our form for the statistical operator $\mathcal{U}$ by supplying a constant C in front, i.e., we assume the form

$$\mathcal{U} = C\,\omega \exp(Q^{+})\,\mathcal{V} \exp(Q)\,\omega, \tag{6·4}$$

where $\mathcal{V}$ is defined by (2·6), and Q is defined by (5·8) and earlier equations in section 5.

Clearly we need a criterion to allow us to choose the matrices U_s in some "best" way; this section is devoted to writing down such a criterion, and giving some discussion of it.

A reasonable criterion can be obtained by observing that the standard grand canonical distribution function(*)

$$\mathcal{U} = \exp \beta(\Omega - H + \mu\mathcal{N}) \tag{6·5}$$

is the solution of the *variation principle:*

$$\text{Trace}\,(\mathcal{U}\ \ln\ \mathcal{U}) = \text{Minimum} \tag{6·6}$$

subject to the conditions (6·1), (6·2), and (6·3). The parameters β, μ, and Ω appear as Lagrange multipliers in this variation. The Trace $(\mathcal{U}\ \ln\ \mathcal{U})$ is of course proportional to the negative of the entropy of the system.

Let us now observe that condition (6·6) can be imposed on the approximate density matrix $\mathcal{U}$, equation (6·4). That is, suppose we stop with correlations of order s. We then have s unknown matrices $U_1, U_2, \cdots, U_s$, and the unknown constant C in (6·4). We get conditions on these matrices and on C by imposing (6·6). Of course, the correct solution, (6·5), is not included in our Ansatz (unless we include all orders s of correlations); hence we expect to get self-consistent solutions, rather than exact expansions.

To be specific, consider the *independent particle model*, which is obtained from (6·4) by omitting all $s \geq 2$. We have

$$\mathcal{U} = C\,\mathcal{V} = C \prod_k (\nu_k)^{\alpha_k + \alpha_k} \quad \text{(Independent particle model)}. \tag{6·7}$$

The parameters available for variation are the set of numbers ν_k and the one number C.

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* We assume that the Hamiltonian H and the number operator commute with each other.

The actual calculation has been performed by Husimi; the results are as follows: let the Hamiltonian H consist of single terms and pair terms:

$$H=\sum_k h_k a_k^+ a_k+\tfrac{1}{2}\sum_{k_1k_2k_1'k_2'}\langle k_1k_2|v|k_1'k_2'\rangle a_{k_2}^+ a_{k_1}^+ a_{k_1'} a_{k_2'} \tag{6·8}$$

and define:

$$\bar{n}_k\equiv u_k/(1+u_k) \tag{6·9}$$

$$V_k\equiv\sum_{k'}[\langle kk'|v|kk'\rangle-\langle kk'|v|k'k\rangle]\bar{n}_{k'}. \tag{6·10}$$

The first is the average number of particles in state k, the second the average potential energy encountered by a particle in state k. When we perform the variational calculation, we introduce Lagrange multipliers involving β and μ, and arrive at the self-consistent *Husimi equations*:

$$u_k=\exp[\beta(\mu-h_k-V_k)], \tag{6·11}$$

$$N=\sum_k \bar{n}_k, \tag{6·12}$$

$$E=\sum_k(h_k+\tfrac{1}{2}V_k)\bar{n}_k. \tag{6·13}$$

Equations (6·9)—(6·13) form a self-consistent set, to determine the quantities β, μ, and V_k, given E, N, and the Hamiltonian (6·8). In the limit of zero temperature, these equations reduce to the Hartree-Fock equations (6·11) becoming a step-function distribution.

We have written down these equations explicitly to show an important difference between the self-consistent equations for the ground state and for statistical mechanics. For the ground state, the Hartree-Fock equations of an infinite system become trivial, i.e., the self-consistent wave functions are known to be plane waves (from translation symmetry) and the answer for the energy follows at once. In statistical mechanics, even this simplest case leads to a non-linear integral equation: equation (6·10) involves V_k on the left side, and a sum over quantities containing $V_{k'}$ on the right side.

In principle, we can now extend the method to include pair correlations (the quasi-chemical equilibrium theory) and even higher correlations. We replace assumption (6·7) for the density matrix by (6·4), and again perform the variation (6·6) subject to (6·1)—(6·3). However, the evaluation of the trace (6·6) now becomes considerably more complicated, and the calculation is still in progress.

We therefore restrict ourselves to some remarks on matters of principle. First of all, the traces (6·1), (6·2), and (6·6), which are needed for the variational formulation of the theory, all involve *off-diagonal* matrix elements of ζ. Hence the extension of the original theory to include an Ansatz for off-diagonal elements, performed in sections 2 and 5, is an essential first step in the variational approach, and thus necessary for a complete formulation of the theory (one that includes sufficient equations to specify the Ursell matrices themselves).

Secondly, there is an ambiguity inherent in the definition of the Ursell matrices

for $s \geq 2$. For example, consider a "pair correlation" matrix $\langle k_1 k_2 | U_2 | k_1' k_2' \rangle$. We can add to this matrix a direct product of the "single-particle" matrices U_1, multiplied by some constant c, that is we construct the matrix U'_2:

$$\langle k_1 k_2 | U_2' | k_1' k_2' \rangle = \langle k_1 k_2 | U_2 | k_1' k_2' \rangle + c \langle k_1 | U_1 | k_1' \rangle \langle k_2 | U_1 | k_2' \rangle. \tag{6·14}$$

The added term is not really a "pair correlation" at all, and it is therefore necessary to find a condition for determining the constant c uniquely. Now, in the simplest form of the theory, the matrices U_1 and U_2 are given by equations (1·3) and (1·4); furthermore, with this choice of U_2, the trace of U_2 is proportional to the second virial coefficient of the dilute gas:[1)]

$$\text{Trace } U_2 = \sum_{k_1 k_2} \langle k_1 k_2 | U_2 | k_1 k_2 \rangle = -\frac{V}{\lambda^6} b, \tag{6·15}$$

where V is the volume, λ is the usual thermal wave length,

$$\lambda = \left[\frac{4\pi\hbar^2}{2mkT} \right]^{1/2}, \tag{6·16}$$

and b is the second virial coefficient; this coefficient is an intensive quantity (see equation (2·17) of reference 1)), *not* proportional to the volume. Furthermore, this property of U_2, which we may also consider as a "finite range of the correlation", turned out to be essential for the later discussion (see section 5 of reference 1)).

It is natural, therefore, to fix the coefficient "c" in (6·14) for the general case by the same condition, i.e., we require

$$\text{Trace } U_2 = \text{proportional to volume } V. \tag{6·17}$$

The same condition should be imposed on the traces of U_s for all orders s of correlations.

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Letters to the Editor

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Meson-Meson Interaction

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July 5, 1958

Strong resonant interactions between the incident meson and the meson-cloud of the nucleon were suggested phenomenologically by Dyson[1] and Takeda[2] as a possible interprtation for the second maximum in π^--p scattering in the Bev-region. The former postulates a resonant state in $T=0$, while the $T=1$ state resonance is proposed by the latter.

In this letter we wish to discuss if such a resonance could be explained at least qualitatively from the field theoretical point of view. Such discussion has already been tried by B. Bosco and R. Stroffolini[3] using a simplified hamiltonian. However, their treatment seems unsatisfactory because of the specialized form of hamiltonian. In order to investigate qualitatively whether meson-meson potential is attractive or repulsive near 1 Bev, a perturbation treatment seems to be sufficient. Hence we start with an evaluation of the lowest order S matrix for π-π scattering by referring to Fig. 1, using $ps(ps)$ meson theory. The matrix element can be expanded in a power series in meson momenta k_i if $k\ m \ll 1$ (m being the nucleon mass). The first term is proportional to φ_i^4 with diverging coefficient. This divergence is cancelled by a counter term $\delta\lambda\varphi_i^4$ leaving a finite term $\lambda\varphi_i^4$. This λ is an arbitrary constant and cannot be calculated from known constants.

The second term, proportional to k_ik_j, is without any uncertainty and is expressed as

$$M_{ijkl}(k_1, k_2, k_4) = -i\left(\frac{g^2}{4\pi}\right)^2 \frac{8(2\pi)^4}{3m^2} \times [\delta_{jk}\delta_{il}(-3k_1k_2 - k_1k_4 + 3k_2k_4 - k_1^2 - 3k_2^2 - k_4^2) + \delta_{ij}\delta_{lk}(-3k_1k_2 + 3k_1k_4 - k_2k_4$$

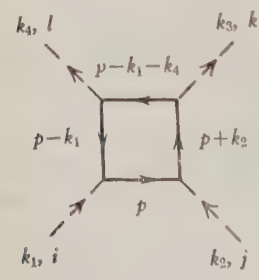

Fig. 1a.

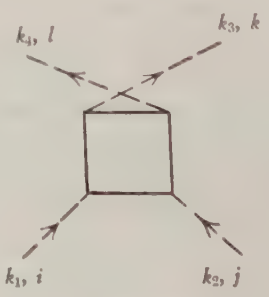

Fig. 1b.

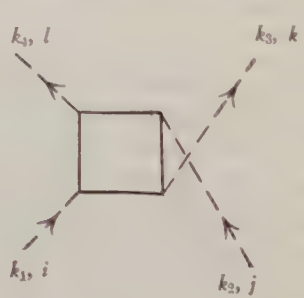

Fig. 1c.

$$-3k_1^2-k_2^2-k_4^2)$$
$$+\delta_{kl}\delta_{ij}(k_1k_2+3k_1k_4+3k_2k_4$$
$$-k_1^2-k_2^2-3k_4^2)]. \quad (1)$$

With the use of the relation between the S-matrix element and the effective hamiltonian V

$$M_{ijkl}(k_1k_2k_3k_4)=-i(2\pi)^4V, \quad (2)$$

the π-π potential can be obtained. Transforming the 4-momenta to the c.m.s.

$$k_1\to(\boldsymbol{p}, p_0),\ k_2\to(-\boldsymbol{p}, p_0),$$
$$k_3\to(-\boldsymbol{q}, p_0),\ k_4\to(\boldsymbol{q}, p_0), \quad (3)$$

the effective hamiltonian is expressed in a simple form when two pions are free.

$$V=\left(\frac{g^2}{4\pi}\right)^2\frac{8}{3m^2}\varphi\varphi\varphi\varphi$$
$$\times[\delta_{jk}\delta_{il}\{-4\boldsymbol{p}\cdot\boldsymbol{q}+3\boldsymbol{p}^2+p_0^2\}$$
$$+\delta_{jl}\delta_{ik}\{4\boldsymbol{p}\cdot\boldsymbol{q}+3\boldsymbol{p}^2+p_0^2\}$$
$$+\delta_{kl}\delta_{ij}\{-\boldsymbol{p}^2-7p_0^2\}], \quad (4)$$

where constant terms are neglected in the above expression since this is always associated with an arbitrary constant which should be fitted to the experiment.

The sign of the π-π potential can be investigated for the possible total isotopic spin $T=0$, 1 and 2 as follows.

I) $T=2$ state. In this case, eq. (4) is expressed as

$$V=\left(\frac{g^2}{4\pi}\right)^2\frac{8}{3m^2}\{6\boldsymbol{p}^2+2p_0^2\}\varphi\varphi\varphi\varphi. \quad (5)$$

Now according to the Bose-Einstein statistic, even and odd total isotopic spin states should be associated with even and odd angular momentum states. So, if we calculate the matrix element of the effective hamiltonian V between two symmetric spatial wave functions, the potential proves to be repulsive.

Even if we consider $\lambda\varphi^4$ interaction in addition to this potential, a repulsive potential would be obtained because $\lambda>0$. This is consistent with phenomenological analysis.

II) $T=1$ state. Eq. (4) is expressed in a similar manner as

$$V=-\left(\frac{g^2}{4\pi}\right)^2\frac{8}{3m^2}\{+8\boldsymbol{p}\cdot\boldsymbol{q}\}\varphi\varphi\varphi\varphi. \quad (6)$$

In this case the matrix element of (6) between anti-symmetric spatial wave functions informs us the potential to be attractive. The conclusion is independent of $\lambda\varphi^4$ interaction because p-wave should be associated with $T=1$ state.

III) $T=0$ state. Also in this case eq. (4) is expressed as

$$V=-\left(\frac{g^2}{4\pi}\right)^2\frac{8}{9m^2}\{17\boldsymbol{p}^2+p_0^2\}\varphi\varphi\varphi\varphi. \quad (7)$$

Similarly, the matrix element of (7) becomes negative; however, this does not immediately lead to attractive potential because the $\lambda\varphi^4$ interaction contribute to be positive.

We cannot, of course, conclude only from the above analysis, that the resonance between pions actually occurs in the states of $T=1$ and $T=0$. But the possibility exists that both the states of $T=1$ and $T=0$ contribute to the second maximum in the $\pi-p$ interaction.

According to the investigation by Bosco and Stroffolini, the pion-pion potential is attractive for $T=0$ and repulsive for $T=2$ which is consistent with our results but no information can be obtained for $T=1$.

The authors would like to express their deep gratitudes to Dr. H. Miyazawa for

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Baryon Mass Spectrum

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In our previous report[1], the possibility was pointed out that if the relative parity between nucleon and Ξ particle is assumed to be odd, the lowest order self energies of baryons coming from the charge independent strong interaction give the observed baryon mass level ordering. Schwinger also suggested such a level splitting in his discussion of the isospace symmetry[2]. In the present report, it is shown that the numerical estimate of mass levels in the lowest order approximation is not inconsistent with the observation, when the reasonable magnitudes of coupling constant g_K and of cutoff momentum are used.

Table 1

	N	Λ	Σ	Ξ	K
Case (a)	+	+	+	−	+(direct) and/or −(derivative)
	+	−	−	−	−(direct) and/or +(derivative)
Case (b)	+	−	+	−	+(direct) and/or −(derivative)
	+	+	−	−	−(direct) and/or +(derivative)

As was shown in reference 1, the following choices of relative parities and of K coupling type give the observed level ordering with the approximately correct interval ratio.

Here we assume that all of baryon fields transform as either $\psi\to\pm i\gamma_4\psi$ or $\psi\to\pm\gamma_4\psi$, and the K field $\phi_K\to\pm\phi_K$ under the space reflection.

For simplicity, only the direct coupling of the K field is considered. Then the contributions to the self energy of baryon from the scalar and pseudoscalar direct couplings contain the logarithmically divergent integral. We now adopt the straight cutoff technique, $\int_0^K k^2dk\int d\Omega_k\int_{-\infty}^{\infty}dk_0$, where K is the cutoff momentum. The contributions to the self energy are expressed as follows (4π times $F(\cdots)$ in reference 1),

scalar coupling:

$$\frac{M}{2\pi}\left\{-\frac{3}{2}\log(\kappa+\sqrt{\kappa^2-1})-\frac{1}{4}\left[-2\kappa D+(6\lambda^2-\lambda^4)L+\frac{(16\lambda-8\lambda^3+\lambda^5)}{\sqrt{4-\lambda^2}}T\right]\right\}. \tag{1}$$

pseudoscalar coupling:

$$\frac{M}{2\pi}\left\{\frac{1}{2}\log(\kappa+\sqrt{\kappa^2+1})+\frac{1}{4}\left[-2\kappa D+(2\lambda^2-\lambda^4)L-\frac{(4\lambda^3-\lambda^5)}{\sqrt{4-\lambda^2}}T\right]\right\}, \tag{2}$$

where M is the rest mass of the degenerate bare baryon, $\kappa=K/M$, $\lambda=(\text{meson mass})/M$,

$$D=\sqrt{\kappa^2+1}-\sqrt{\kappa^2+\lambda^2},$$

$$L=\log\left(\lambda\cdot\frac{\kappa+\sqrt{\kappa^2+1}}{\kappa+\sqrt{\kappa^2+\lambda^2}}\right),$$

$$T=\tan^{-1}\frac{\sqrt{4-\lambda^2}}{\lambda}$$

$$-2\tan^{-1}\left(2\cdot\sqrt{\frac{\mu-\kappa}{\mu+\kappa}}\cdot\frac{\sqrt{\kappa^2+1}-\mu}{2-\lambda^2}\right)$$

$$-2\tan^{-1}\left(2\cdot\sqrt{\frac{\mu-\kappa}{\mu+\kappa}}\cdot\frac{\sqrt{\kappa^2+\lambda^2}-\mu}{\lambda^2}\right)$$

and $$\mu=\sqrt{\kappa^2+\lambda^2-\frac{\lambda^4}{4}}\,.$$

The surface term, which results from the translation of momentum variable, is given by

$$\frac{M}{2\pi}\left\{\frac{\kappa^3}{4}\left(\frac{1}{\sqrt{\kappa^2+1}}-\frac{1}{\sqrt{\kappa^2+\lambda^2}}\right)-\frac{\kappa^3\,\lambda^2}{16\,\mu^2}\times\left(\frac{2-\lambda^2}{\sqrt{\kappa^2+1}}+\frac{\lambda^2}{\sqrt{\kappa^2+\lambda^2}}\right)\right\} \qquad (3)$$

for the both couplings. This effect, however, does not contribute to the final results of the mass differences. In the limit of $\kappa\to\infty$, the expressions (1), (2) and (3) tend to those given by Enatsu[3].

Now we can obtain the relationship between the coupling constant $g_K^2/4\pi$ and the cutoff parameter κ so as to give the observed mass difference, $M(\Xi)-M(N)=748\,m_e$, using equations (2a) or (2b) in reference 1 and (1) to (3) in this note. The relations in the both cases (a) and in (b) are shown in Fig. 1, for $M=2400m_e$. The numerical value of M does not sensitively affect the mass difference. Fig. 1 shows that if we take $g_K^2/4\pi\sim1$, as is inferred from the recent experiments, we obtain the reasonable magnitudes of cutoff parameter, $\kappa\sim0.8$ and 1.2 for the cases (a) and (b) respectively.

For the case (b) we may further obtain the relationship between $g_\pi^2/4\pi$ and κ so as to give the observed mass difference, $M(\sum)-M(\Lambda)=152\,m_e$, assuming the direct coupling with the π field and using (2b) in reference 1 and (1) to (3) in this note. Fig. 1 shows, however, that

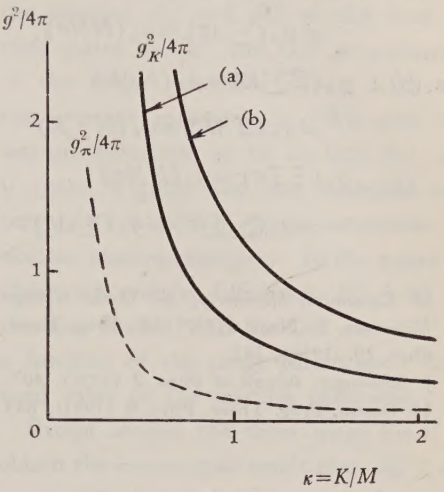

Fig. 1.

$g_\pi^2/4\pi\sim10$ requires too small cutoff parameter. We may ascribe this insufficiency to the largeness of $g_\pi^2/4\pi$, that is, the poor approximation of perturbation method for the π coupling. Otherwise we may regard the $\sum-\Lambda$ mass difference as a higher order effect of the K coupling.

The baryon mass spectra obtained above are shown in Fig. 2 for the both cases.

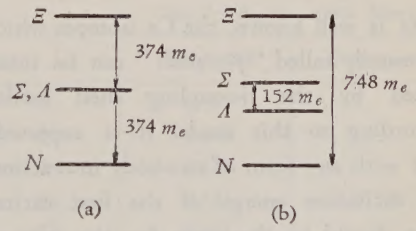

Fig. 2.

In the above discussion, all the K coupling constants and all the π coupling constants are equally taken to be g_K and g_π respectively. The following choices of the coupling constants, however, also give the similar results.

case (a) : $g_K(\Xi\sum\bar{K})=g_K(\Xi\Lambda\bar{K})$
$\neq g_K(N\sum K)=g_K(N\Lambda K)$,

$$g_\pi(\Xi\Xi\pi) = g_\pi(\Sigma\Sigma\pi)$$
$$= g_\pi(\Sigma\Lambda\pi) = g_\pi(NN\pi),$$

case (b) : $g_K(\Xi\Sigma\bar{K}) = g_K(N\Lambda K)$
$$\neq g_K(\Xi\Lambda\bar{K}) = g_K(N\Sigma K)$$
$$g_\pi(\Xi\Xi\pi) = g_\pi(NN\pi)$$
$$\neq g_\pi(\Sigma\Sigma\pi) \neq g_\pi(\Sigma\Lambda\pi).$$

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First Excited State in Ca^{42} and Ca^{44}

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As is well known, the Ca isotopes which are usually called "$f_{7/2}$ shell" can be interpreted by the jj-coupling shell model. According to this model it is expected[1] that with any form of two-body interaction, the excitation energy of the first excited state should be the same for the case in which there are two particles in the $f_{7/2}$ shell as for the case with four particles in the shell. However, the energies of the first excited state in Ca^{42} and Ca^{44} are experimentally different.[2]

Ca^{42}—1.53 Mev
Ca^{44}—1.16 Mev

To interprete this discrepancy we calculated lower levels of Ca^{42} and Ca^{44} by using the method of configuration mixing. The analysis[2,3] of the data in Ca^{41} and the level order of the independent shell model has led us to assume that the following configurations are mixed with the main configuration $\{(f_{7/2})^{n=2,4}J\}$.

Ca^{42}

$J=0$: $\{(f_{5/2})^2 0\}$, $\{(p_{3/2})^2 0\}$

$J=2$: one particle excitation configurations
$\{(f_{7/2}f_{5/2})2\}$, $\{(f_{7/2}p_{3/2})2\}$
two particle excitation configurations
$\{(f_{5/2})^2 2\}$ $\{(p_{3/2})^2 2\}$

Ca^{44}

$J=0$: one particle excitation configurations
$\{(f_{7/2})^3 5/2, f_{5/2}:0\}$, $\{(f_{7/2})^3 3/2, p_{3/2}:0\}$
two particle excitation configurations
$\{(f_{7/2})^2 J_1 (f_{5/2})^2 J_1 : 0\}$, $\{(f_{7/2})^2 J_2 (p_{3/2})^2 J_2 : 0\}$
where $J_1=0$, 2, 4 and $J_2=0$, 2.

$J=2$: one particle excitation configurations
$\{(f_{7/2})^3 J_1 f_{5/2} : 2\}$, $\{(f_{7/2})^3 J_2 p_{3/2} : 2\}$
where $J_1=7/2$, $3/2$, $5/2$, $9/2$ and
$J_2=7/2$, $3/2$, $5/2$.
two particle excitation configurations
$\{(f_{7/2})^2 0, (f_{5/2})^2 2 : 2\}$, $\{(f_{7/2})^2 2, (f_{5/2})^2 0 : 2\}$
$\{(f_{7/2})^2 0, (p_{3/2})^2 2 : 2\}$, $\{(f_{7/2})^2 2, (p_{3/2})^2 0 : 2\}$,
where the sum of the partial seniority numbers was assumed to be limited to 2 to simplify the computation. This assumption is considered to be reasonable.

The two-body interaction in the unfilled shell was assumed to be of the form;

$$G = (\vec{\tau}_1 \cdot \vec{\tau}_2)(1 + a\vec{\sigma}_1 \cdot \vec{\sigma}_2) V(r),$$

where

$$V(r) = V_0 e^{-r^2/r_0^2}.$$

For the unperturbed wave function the harmonic oscillator function[4]

$$\psi = r^{-1} R_{nl}(r)\, Y_{lm}$$

$$R_{nl}(r) = N_{nl}\, e^{-(\nu/2)r^2} r^{l+1} L_{n+l+(1/2)}^{l+(1/2)}(\nu r^2)$$

was used, where R_{nl}, Y_{lm}, N_{nl}, $L_{n+l+(1/2)}^{l+(1/2)}$ have their usual meanings. We also introduced the parameters γ and γ' for the sake of convenience

$$\gamma = \frac{\text{single particle level distance } f_{5/2} - f_{7/2}}{\text{pairing energy of } f_{7/2} \text{ shell}}$$

$$\gamma' = \frac{\text{single particle level distance } p_{3/2} - f_{7/2}}{\text{pairing energy of } f_{7/2} \text{ shell}}.$$

We assumed that γ and γ' have the values given by

$$\gamma = 2\gamma' = 1.4.$$

This numerical value corresponds to the value of 4.2Mev[5] for the $f_{5/2} - f_{7/2}$ separation if the pairing energy in the $f_{7/2}$ shell is assumed to be 3Mev. We computed the energies $\Delta_{02}^{(2)}$ and $\Delta_{02}^{(4)}$ of the first excited states of Ca^{42} and Ca^{44} as a function of the range parameter $\lambda (= \sqrt{\nu}\, r_0)$ and the mixture parameter a. We took the mixture parameter to be $a = 0.2$, 0.1 and 0 (only Wigner) and also Rosenfeld mixture $a = 2.3$ which gives saturation of nuclear binding energies. In the figure we show the energy difference $\Delta_{02}^{(2)} - \Delta_{02}^{(4)}$ of the first excited state in Ca^{42} and Ca^{44} as a function of the range parameter λ for a fixed value of the mixture parameter.

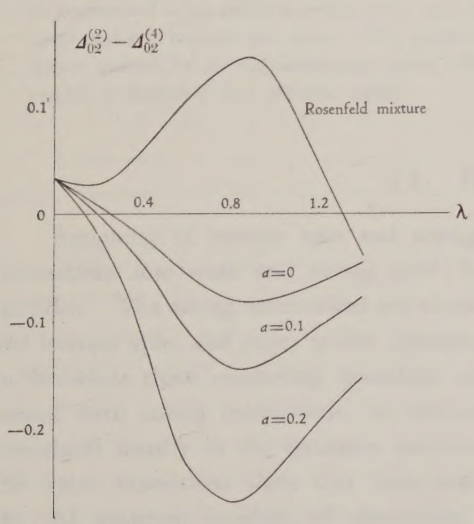

Fig. 1. Difference of the first excitation energies, $\Delta_{02}{}^{(2)} - \Delta_{02}{}^{(4)}$, between Ca^{42} and Ca^{44} in units of $(f_{7/2}{}^2 0 | G | f_{7/2}{}^2 0)$ as a function of the range λ.

Except around the short range limit, we obtain the inconsistent result that $\Delta_{02}^{(2)} < \Delta_{02}^{(4)}$ at $a = 0.2$, 0.1 and 0, but can obtain the desired trend $\Delta_{02}^{(2)} > \Delta_{02}^{(4)}$ for Rosenfeld mixture in the region of the range which we considered. If we tentatively take the pairing energy for the $f_{7/2}$ shell to be 3Mev, we obtain the following numerical values for the energy difference $\Delta_{02}^{(2)} - \Delta_{02}^{(4)}$.

Table 1. Calculated values of the energy difference of the first excited states of Ca^{42} and of Ca^{44} for Rosenfeld mixture. $(f_{7/2}{}^2 0 | G | f_{7/2}{}^2 0) = 3\text{Mev}$.

λ	$\Delta_{02}{}^{(2)} - \Delta_{02}{}^{(4)}$ (unit Mev)	
	Calculated	Experiment
0	0.103	
0.2	0.084	
0.4	0.179	
0.6	0.322	0.37 Mev
0.8	0.423	
1.0	0.380	
1.2	0.137	

The computed values $\Delta_{02}^{(2)} - \Delta_{02}^{(4)}$ fit well with the experimental value, 0.37 Mev, for the region of the range which seems to be suitable.

Details of calculations of the low lying levels in Ca isotopes together with the

discussions about the relative importance of the one and two particle excitation configurations will be reported in the near future.

We wish to thank Profs. Horie and Morinaga for their interest and advice, and also Prof. Yanagawa for helpful discussions.

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